Order Fractionalization

youtube: Piers Coleman

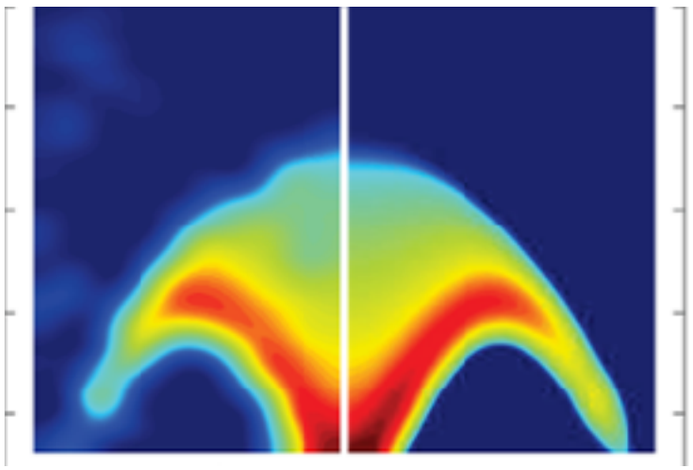
Dynamic Dirac QM

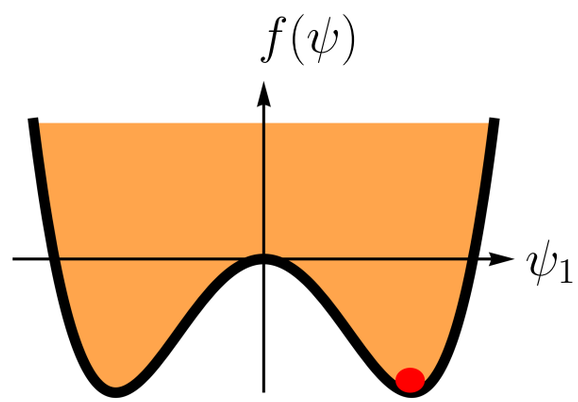
Jacksonville, Fl. 16 Dec 2019

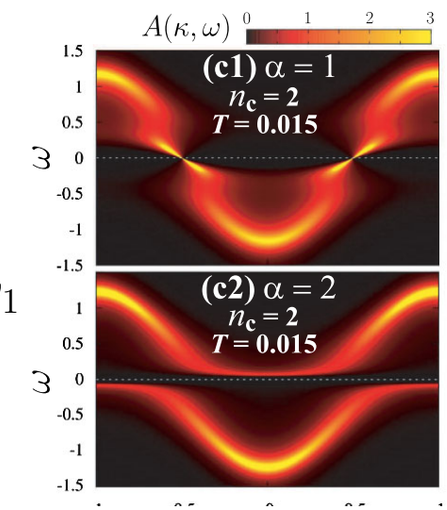
Piers Coleman

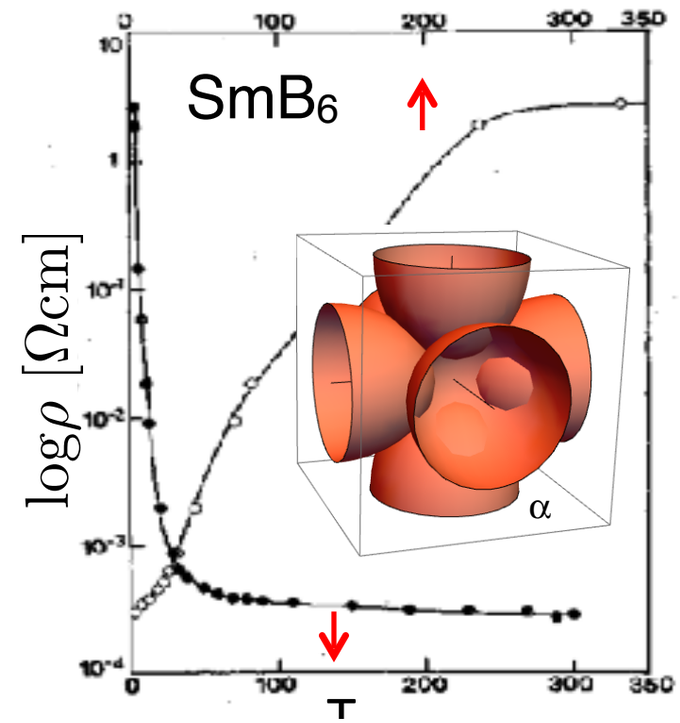
Center for Materials Theory, Rutgers U, USA

Hubbard Theory Consortium, Royal Holloway, U. London









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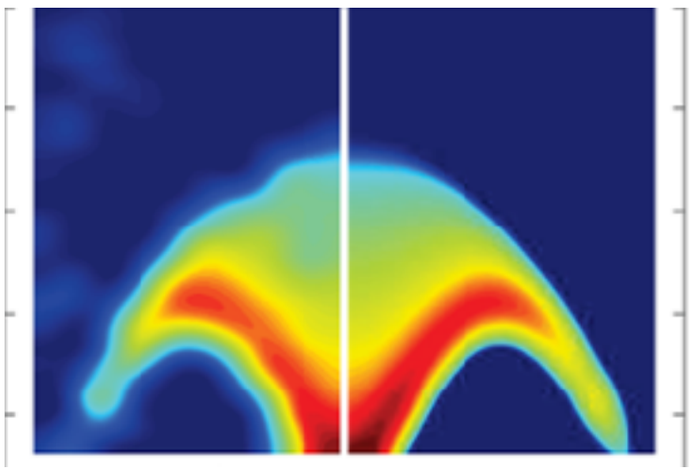

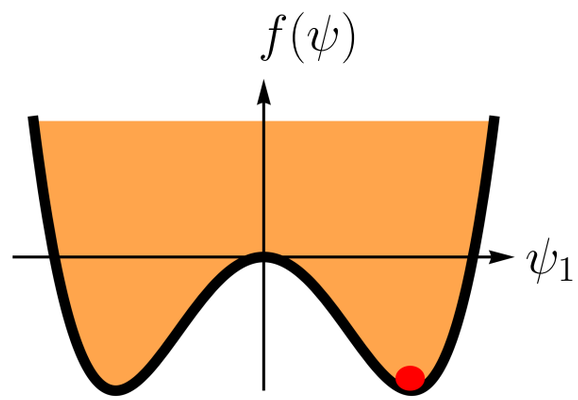

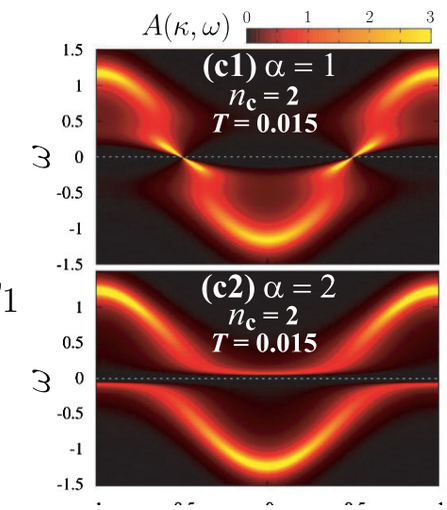

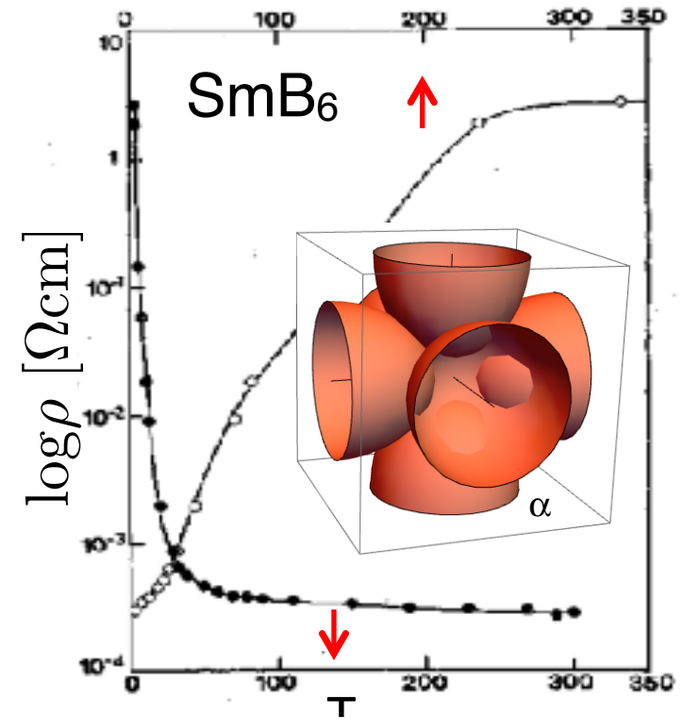

Yashar Komijani (Rutgers),
Anna Toth (Edinburgh),
Premi Chandra (Rutgers)
Ari Wugalter (Rutgers)

arXiv:1811.11115 Y. Komijani, A. Toth, P. Chandra and PC

arXiv:1911.13129 A. Wugalter, Y. Komijani, PC

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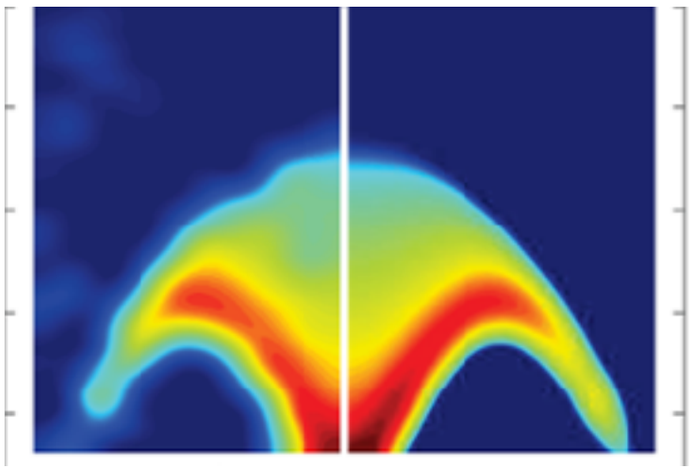

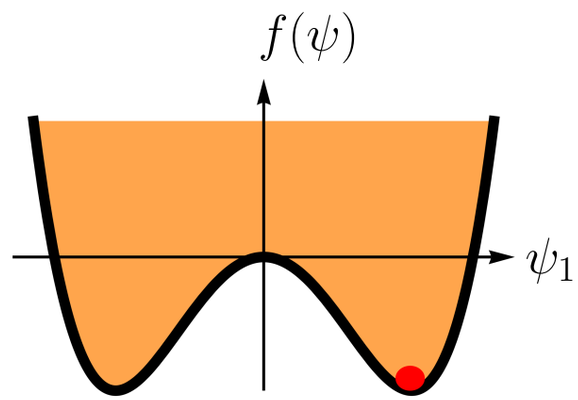

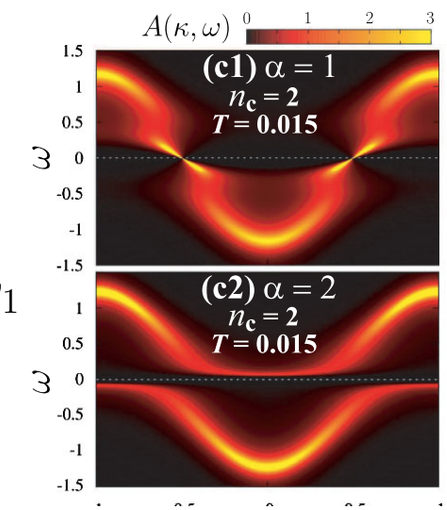

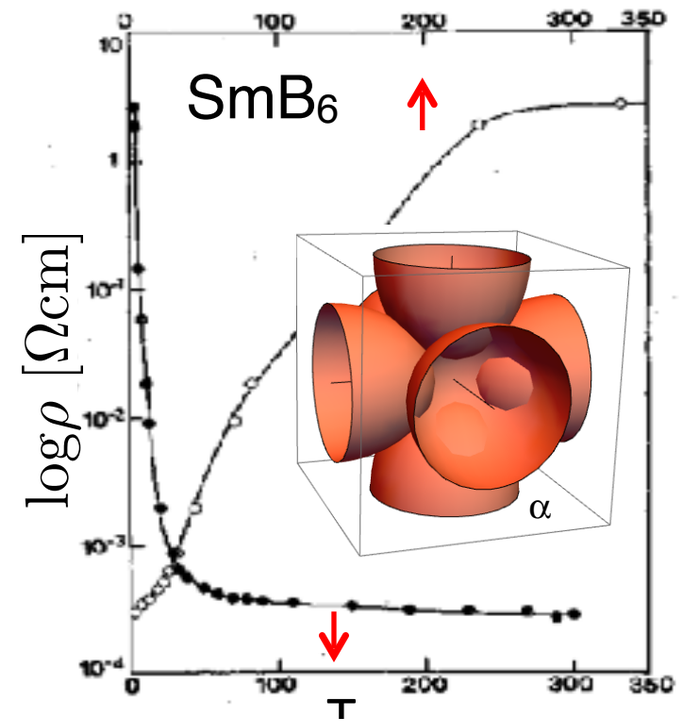

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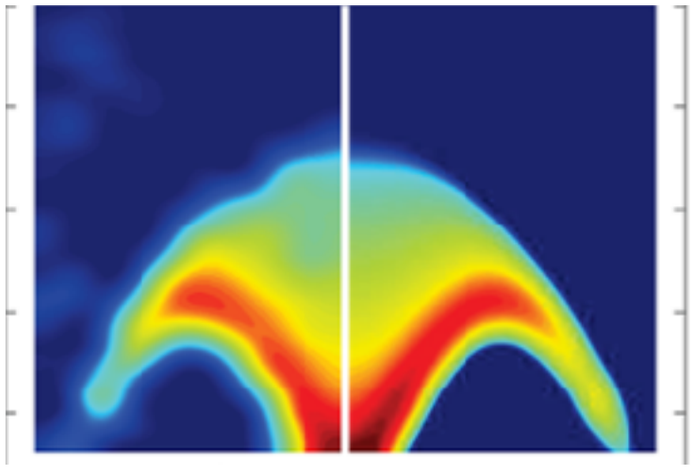

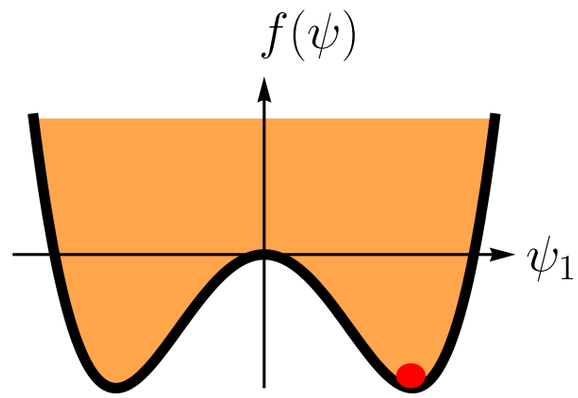

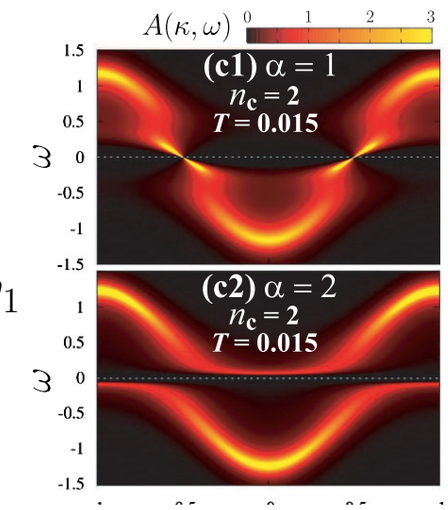

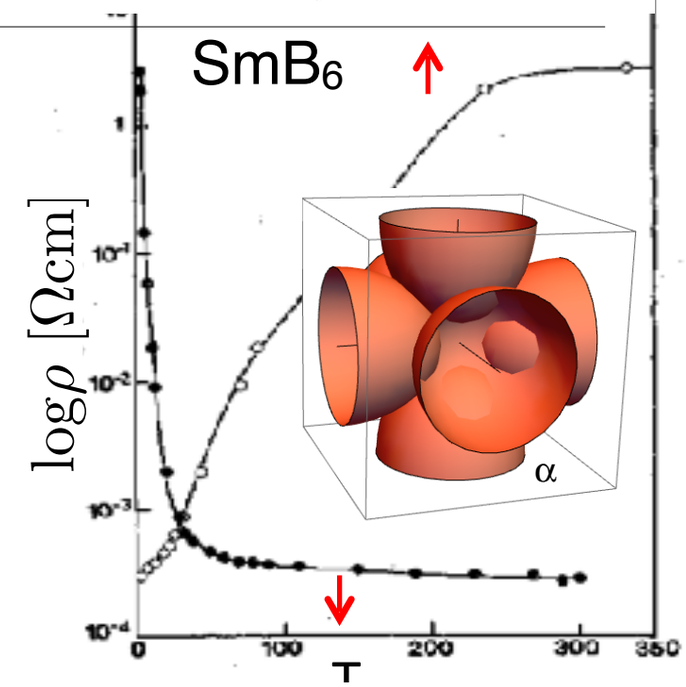

- Fractionalization as Dynamical Order
- Motivation from Experiment
- Induced OFr
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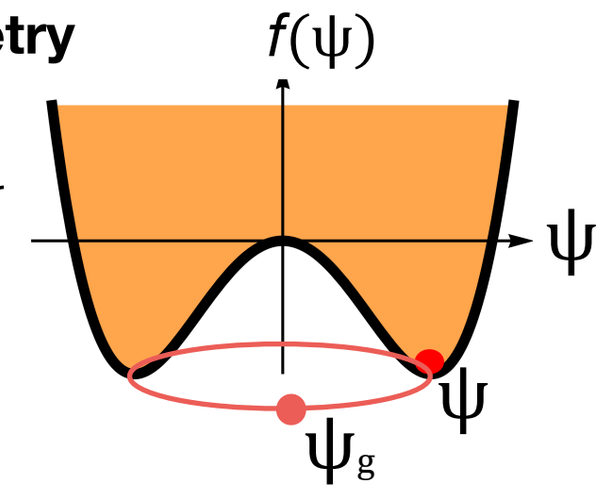
Fractionalization as Dynamic Order

Order and Fractionalization

Order and Fractionalization

Broken Symmetry

Landau (1937)
Order Parameter



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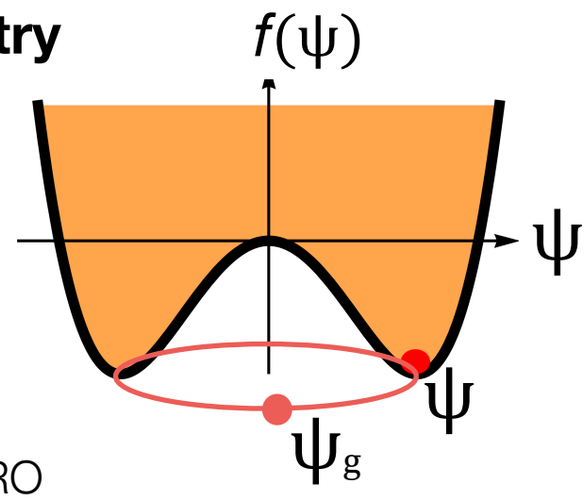

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$$\langle \Psi(x)\Psi^\dagger(y)\rangle \xrightarrow{|x-y|\to\infty} \langle \Psi(x)\rangle\langle \Psi(y)\rangle^*$$

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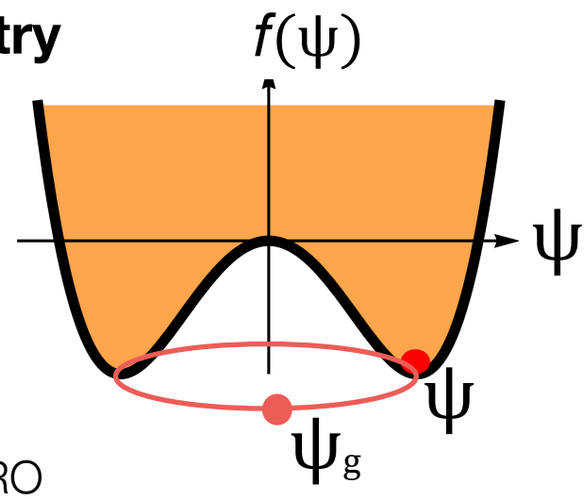

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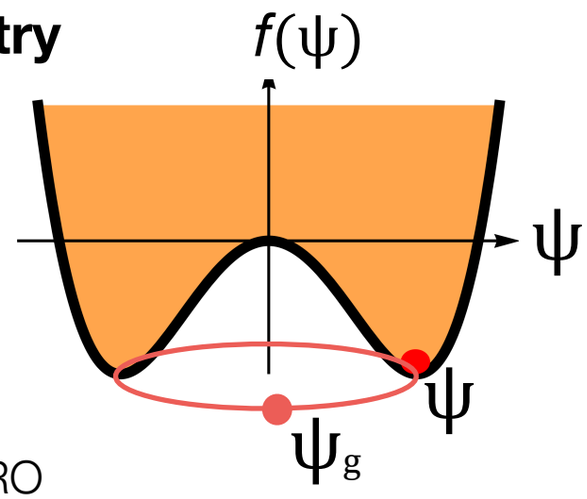

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OPs = pairs of fermions

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BCS

$$\vec{M} = \langle \psi^\dagger \vec{\sigma} \psi \rangle$$

Stoner Hartree Fock

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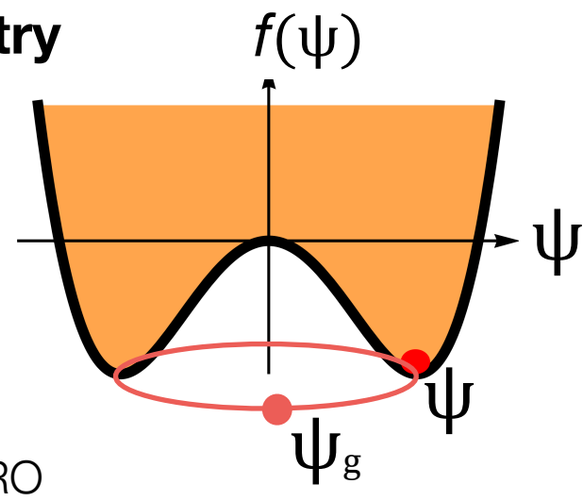

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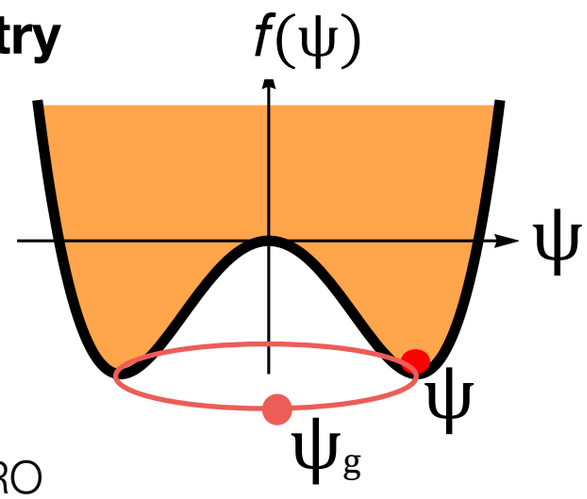

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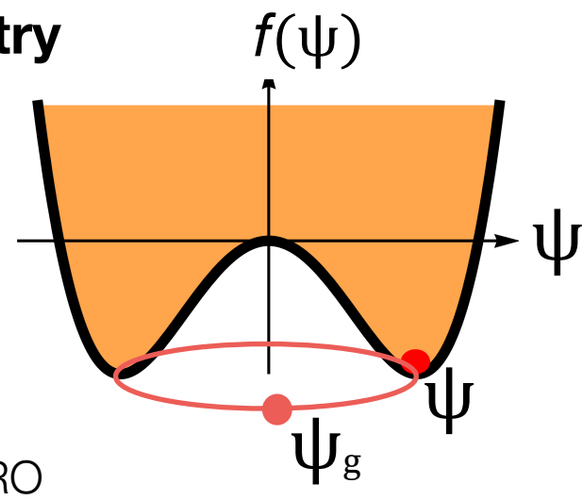

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e.g. Magnons **fractionalize** into Spinons

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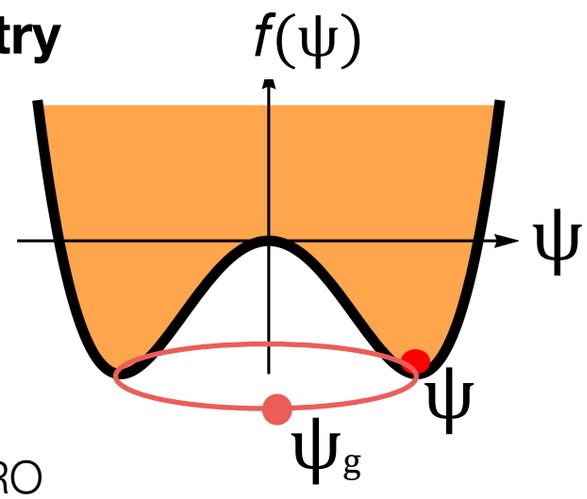

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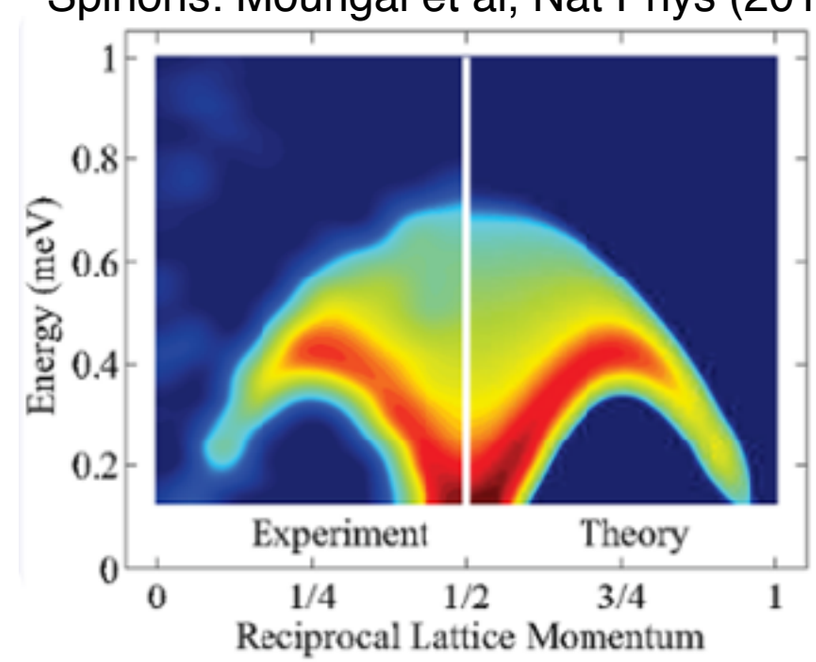

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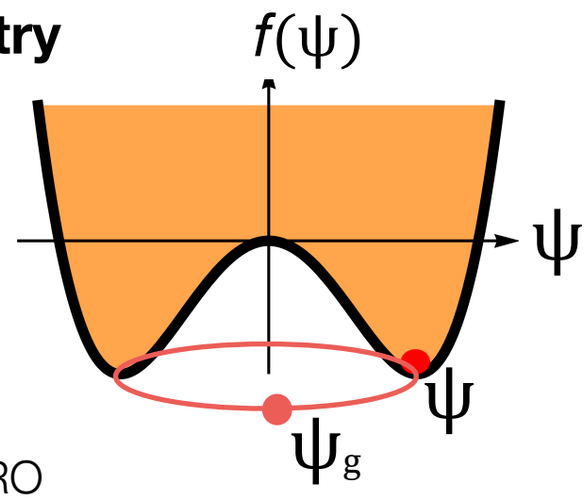

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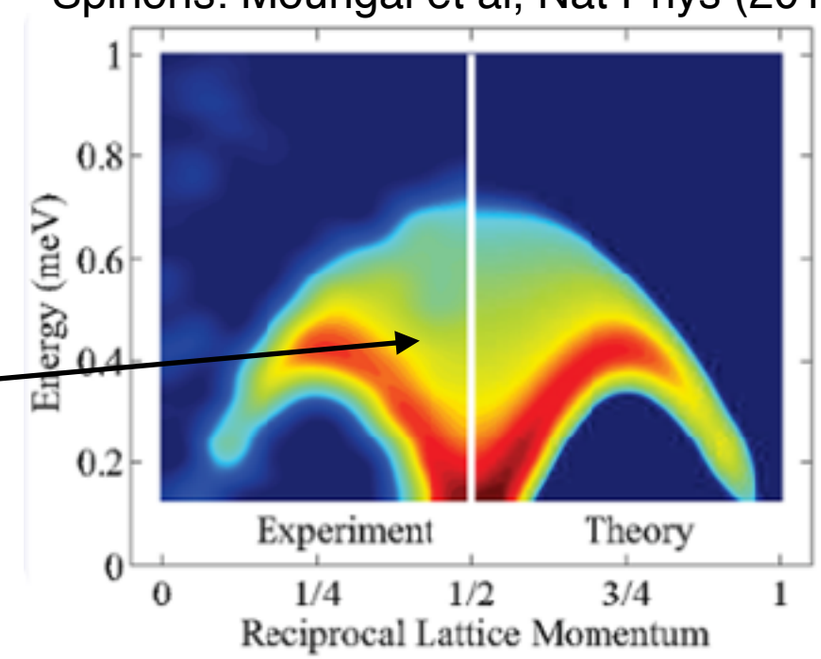

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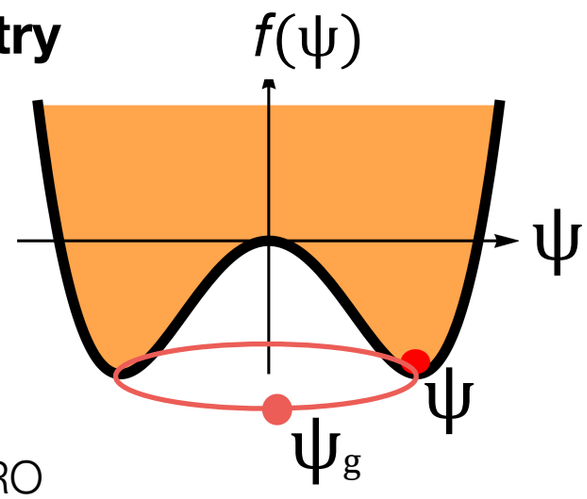

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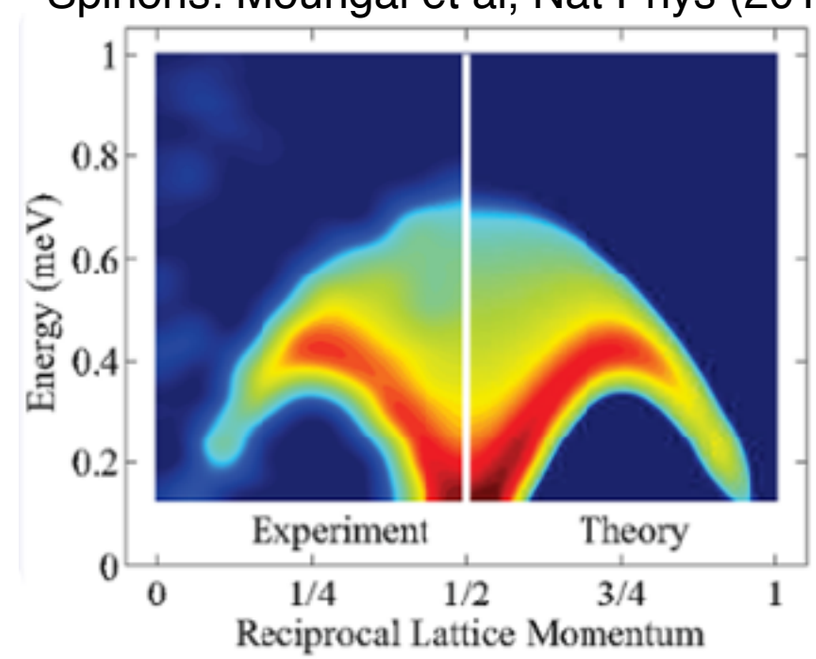

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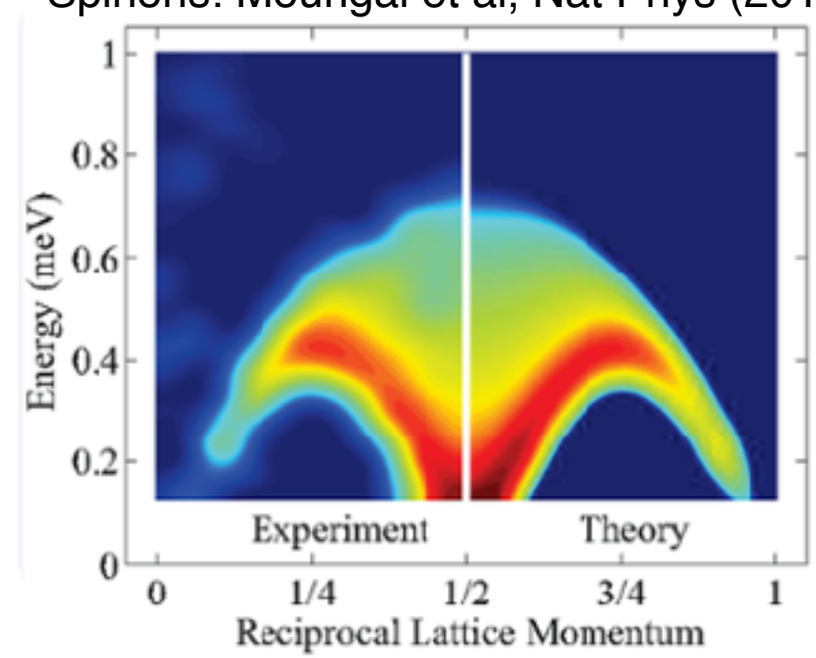
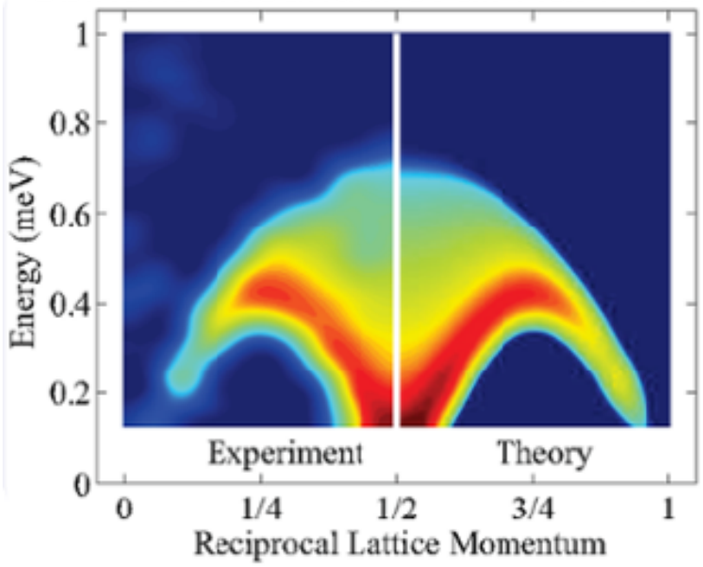


Patterns of Fractionalization

Spinons: Mourigal et al. Nat Phys



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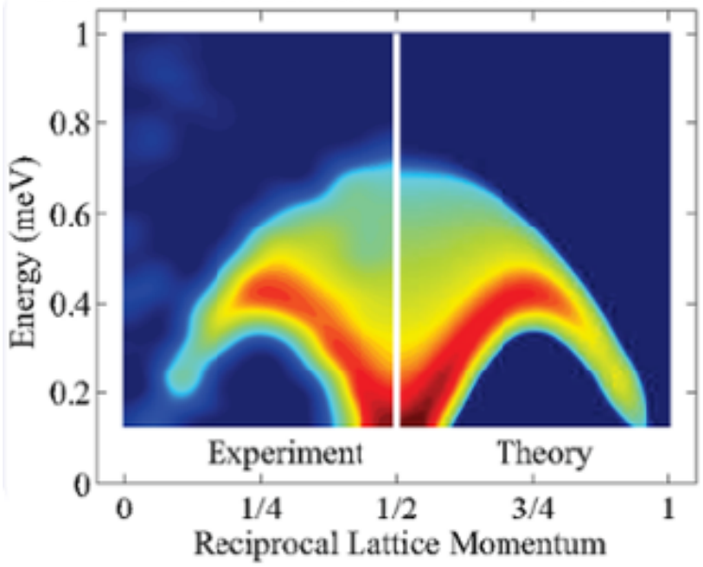

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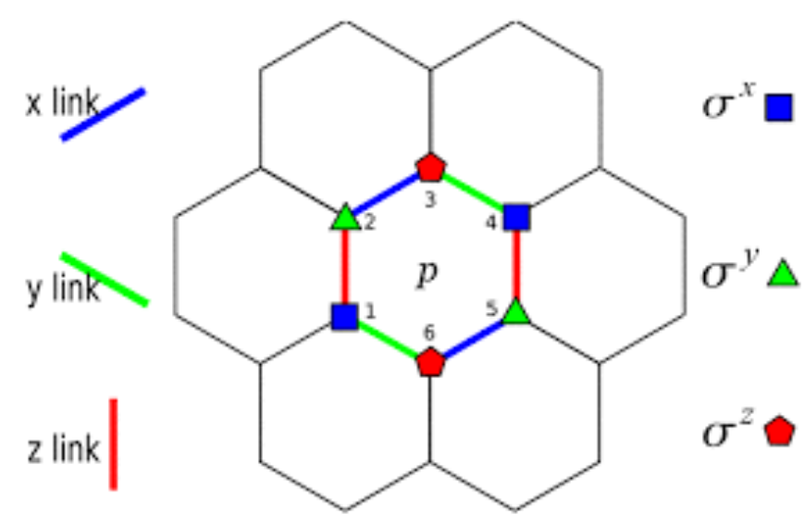

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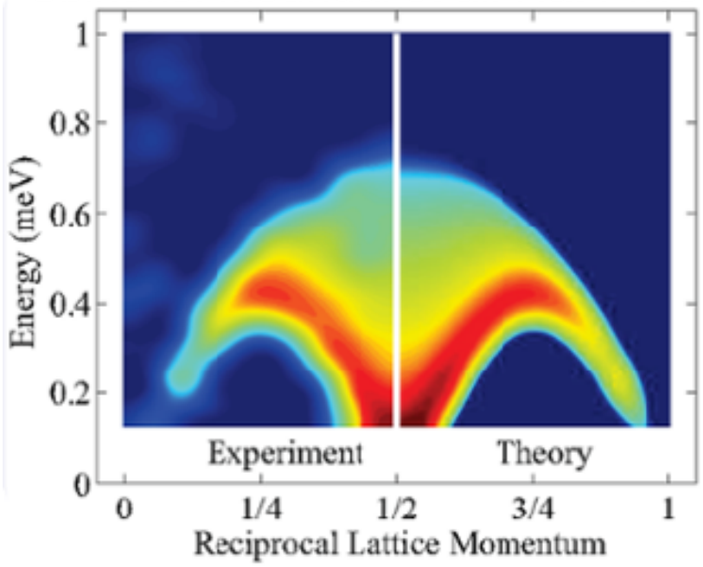

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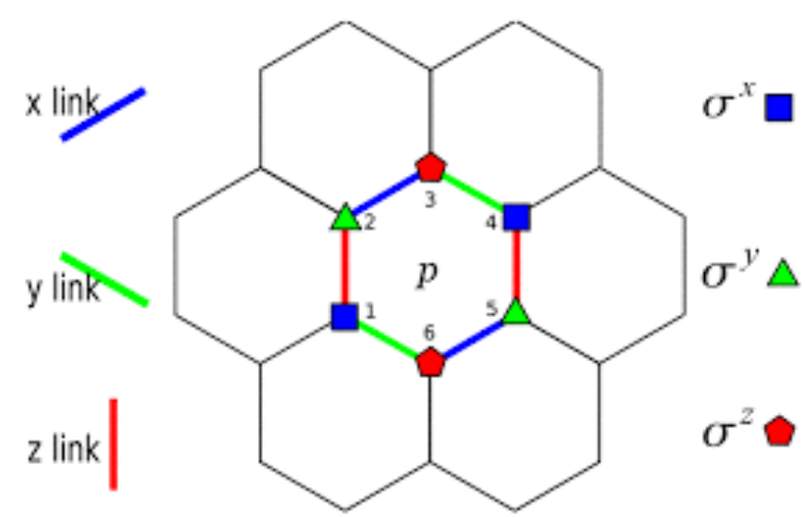

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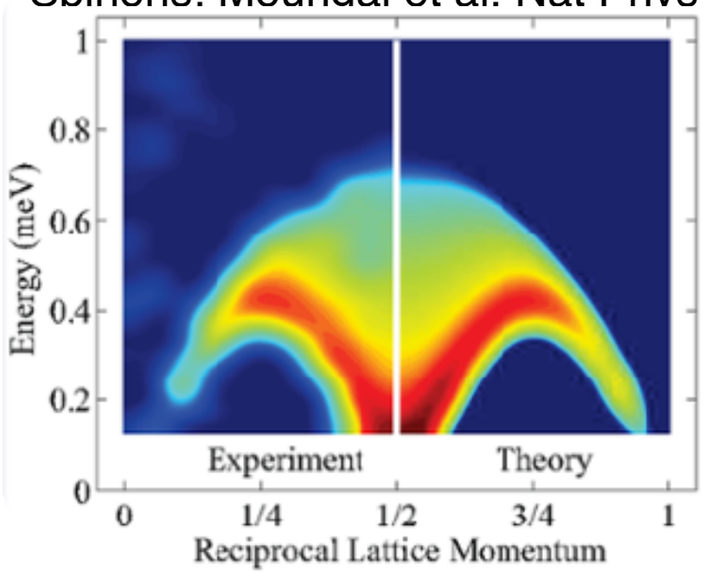

Kamahara et al,
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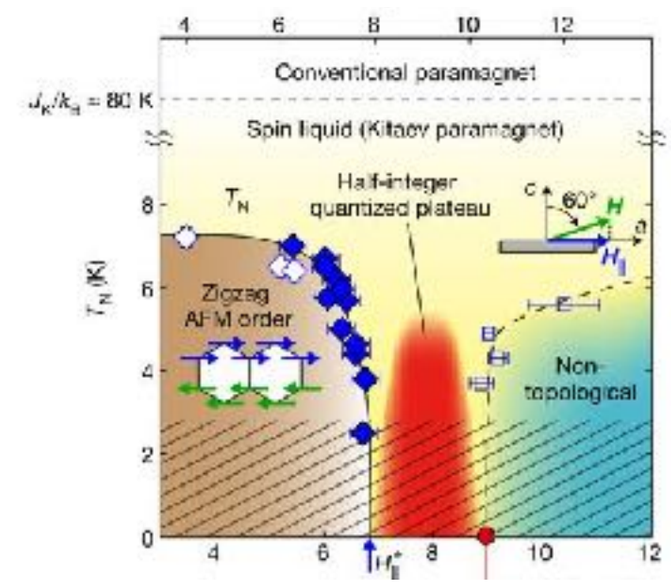

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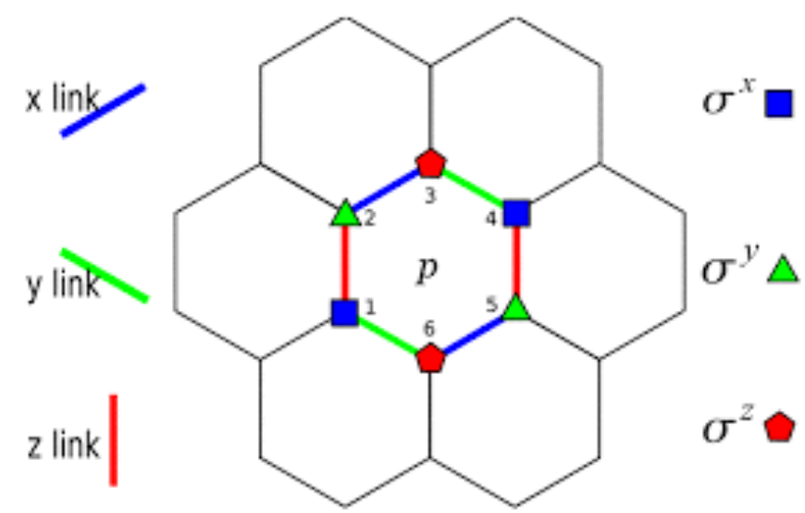

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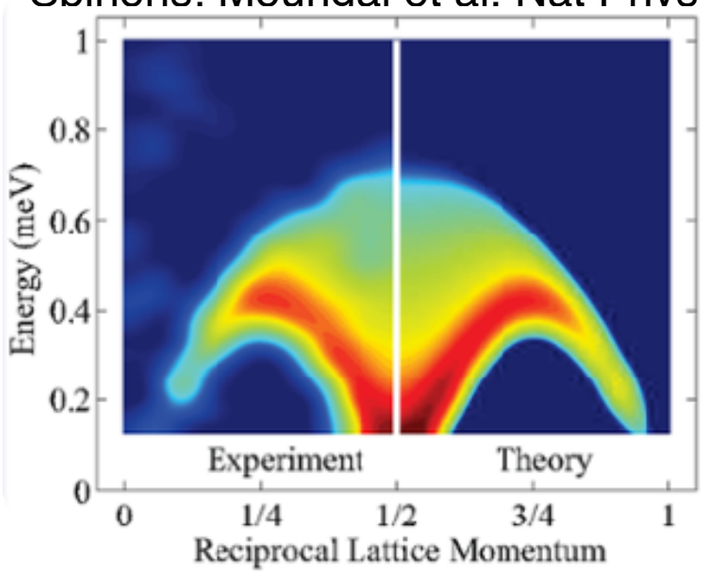

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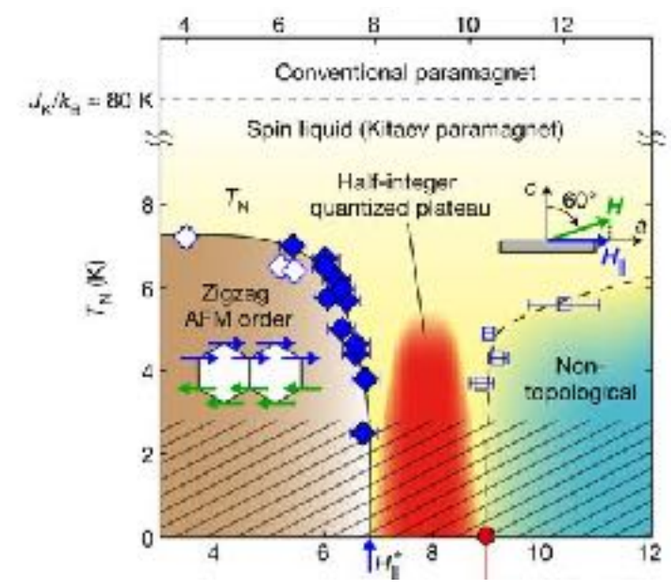

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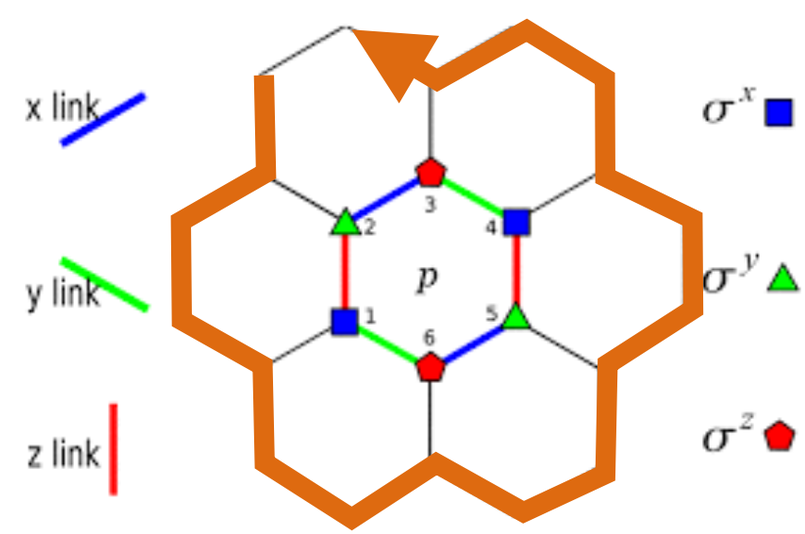

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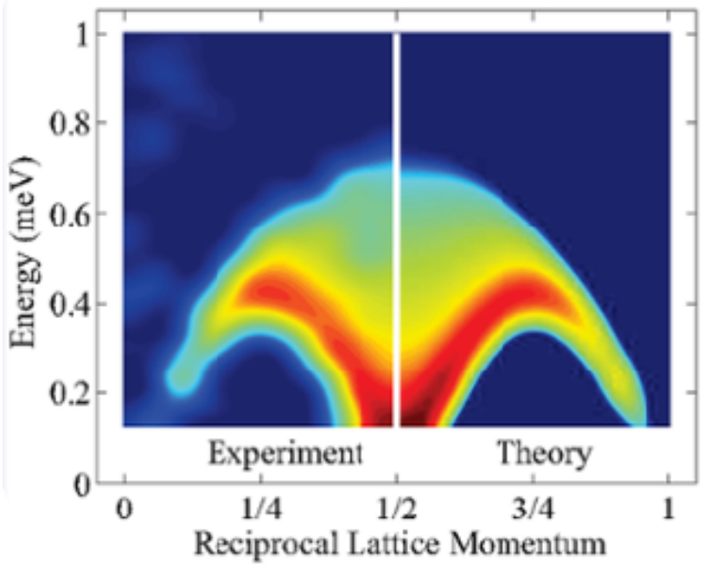

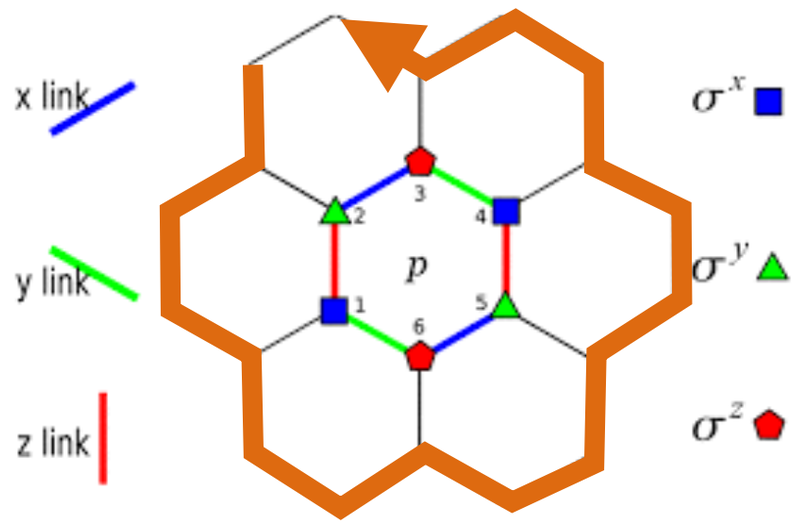

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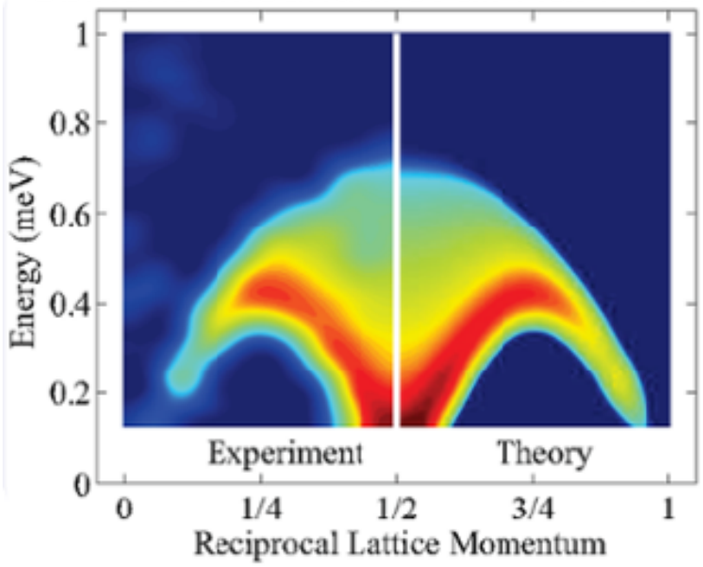
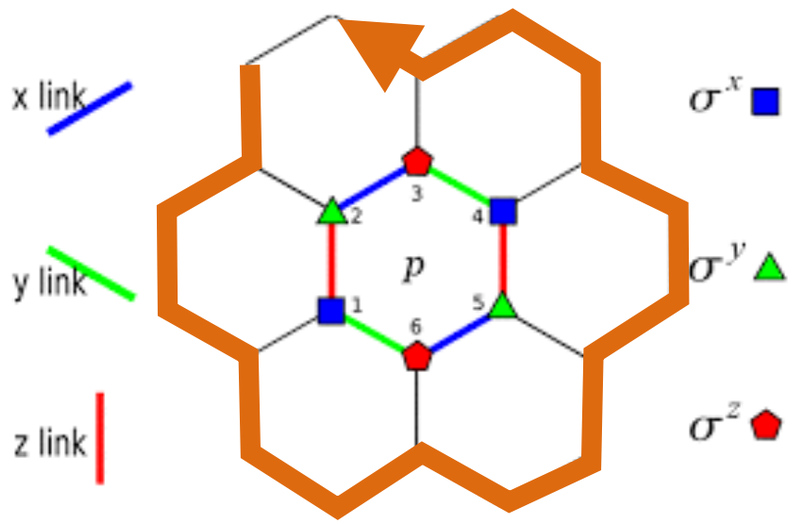


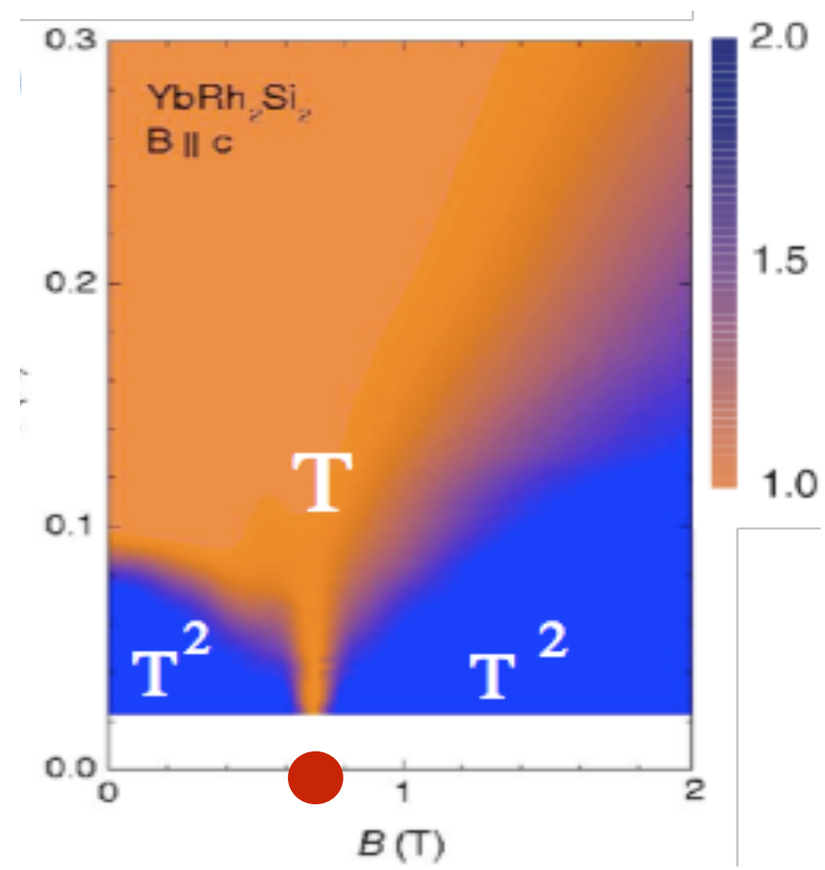


Fractionalization:

S=1/2 Heisenberg Chain,

Kitaev Honeycomb

- *excited* state property
- Dynamic Property



Patterns of Fractionalization

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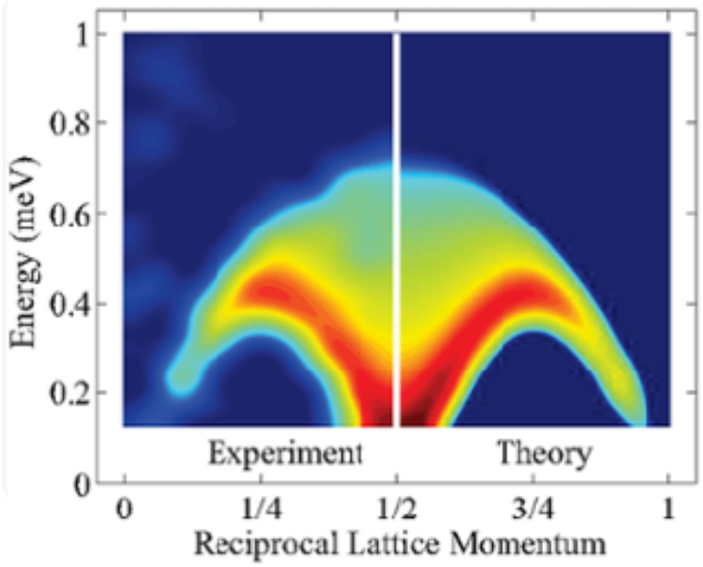

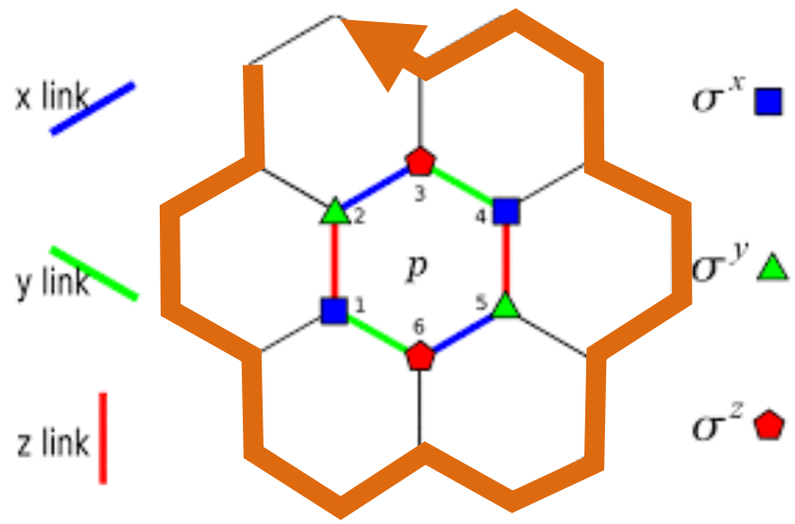

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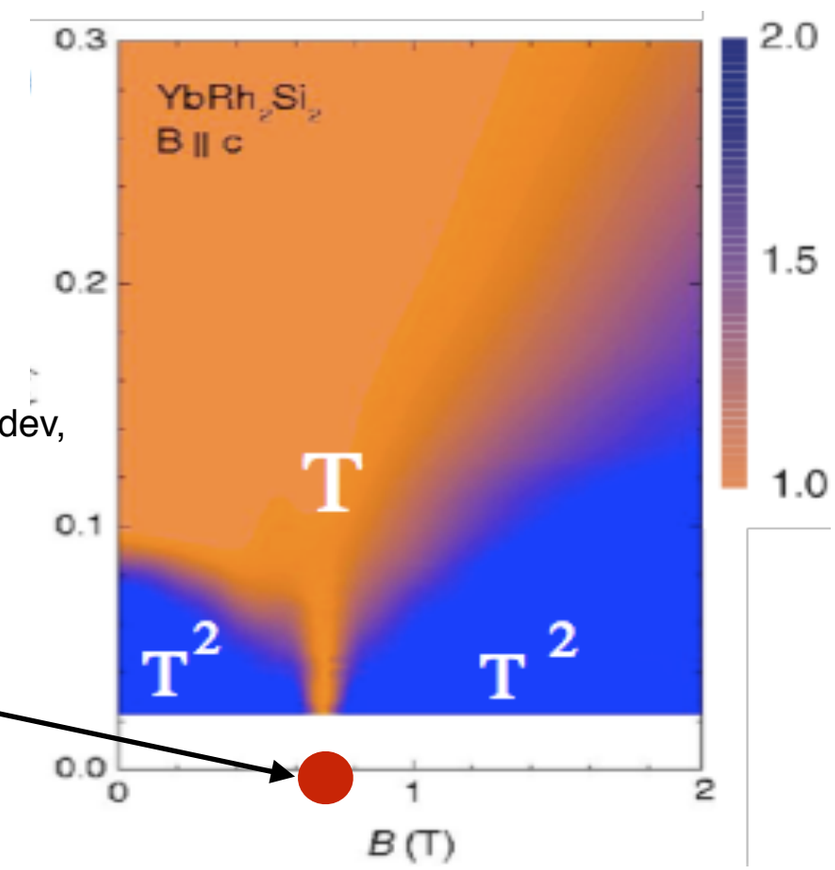
Senthil, Vishwanath, Balents, Sachdev, Fisher. Science, 303, 1490 (2004)

$$\vec{S} \rightarrow b_\alpha^\dagger b_\beta$$

Deconfined Criticality



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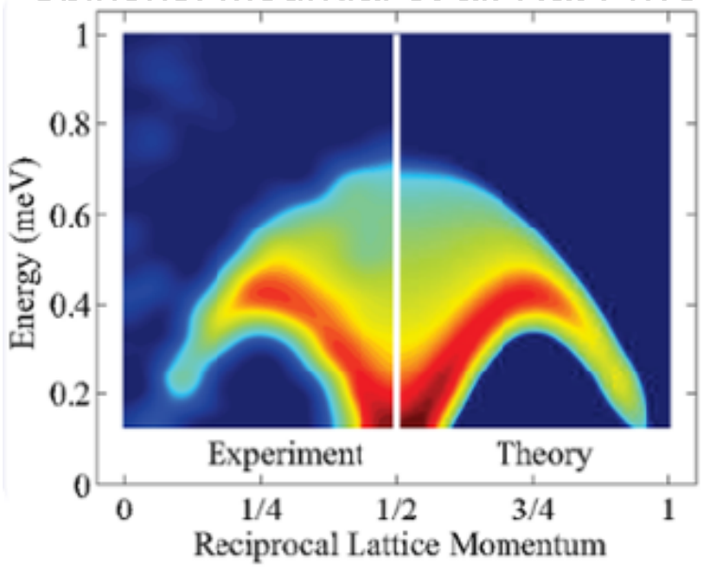

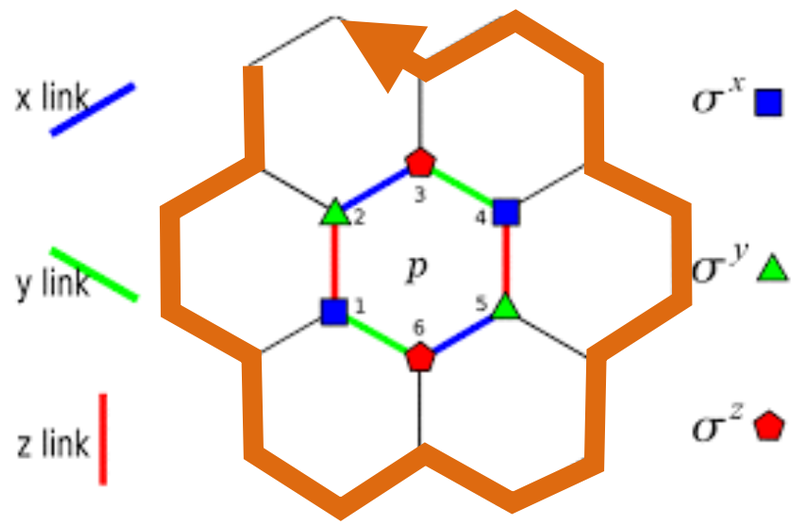

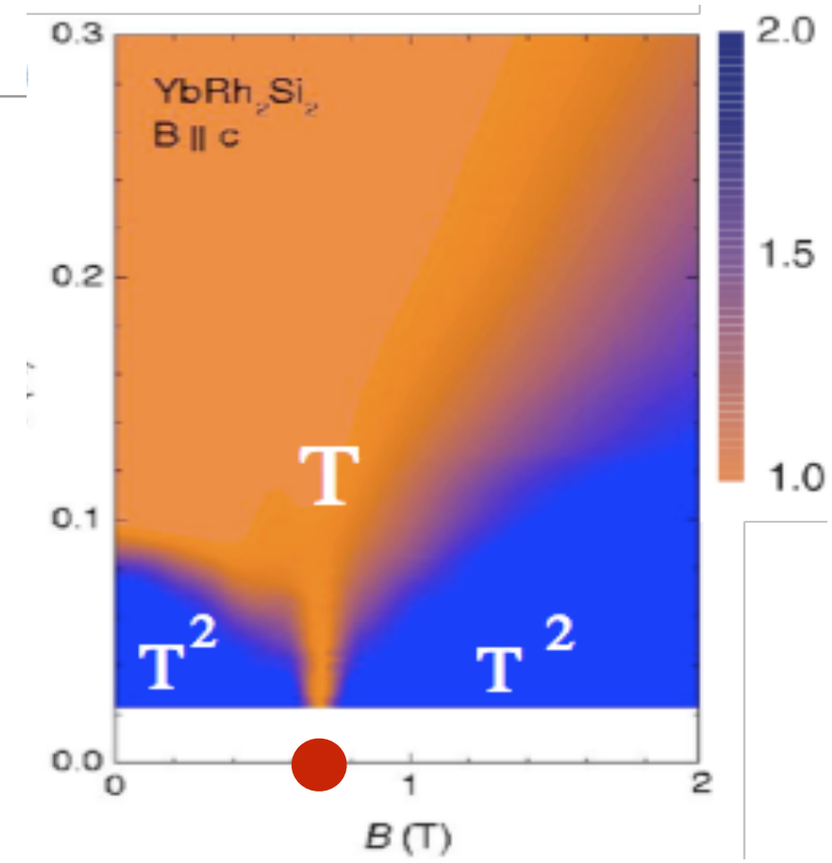

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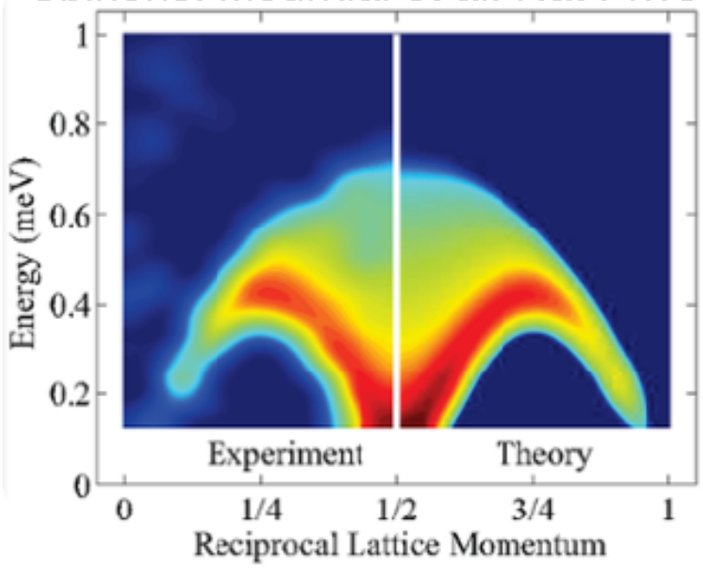

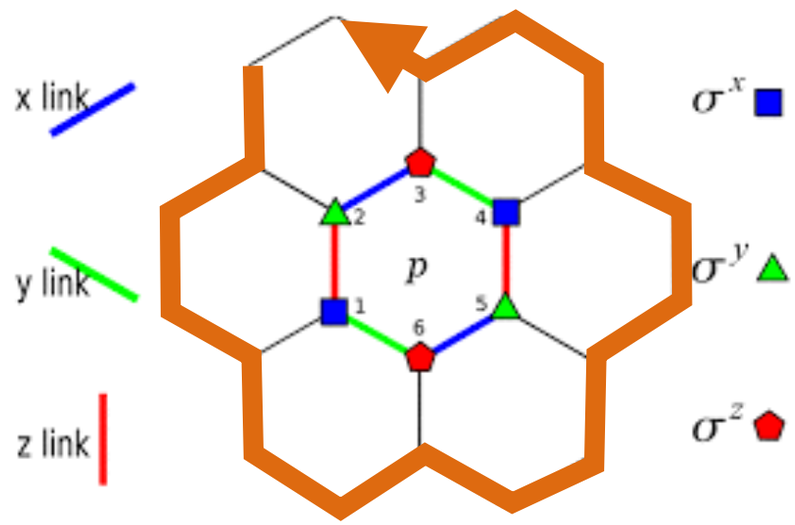

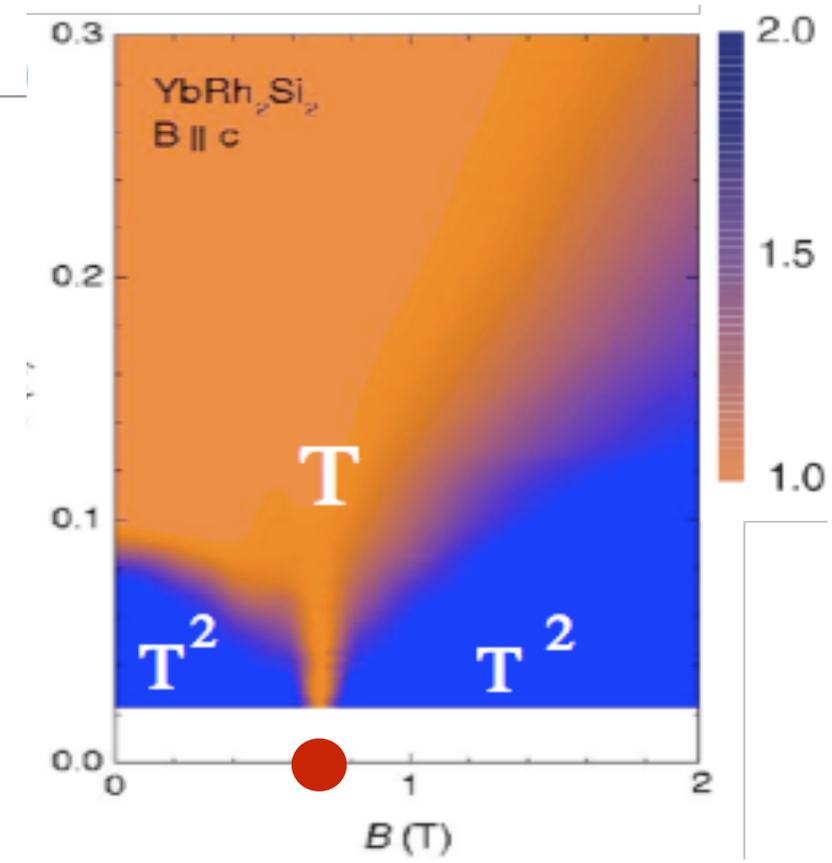

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Many patterns of fractionalization are possible

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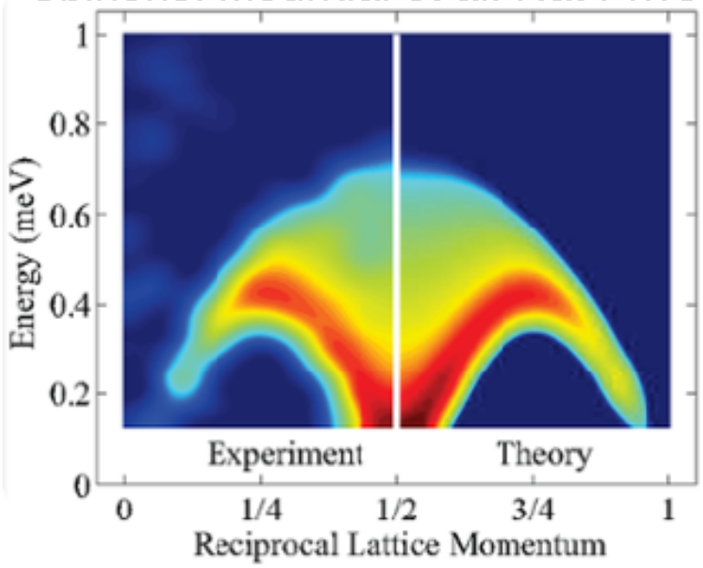

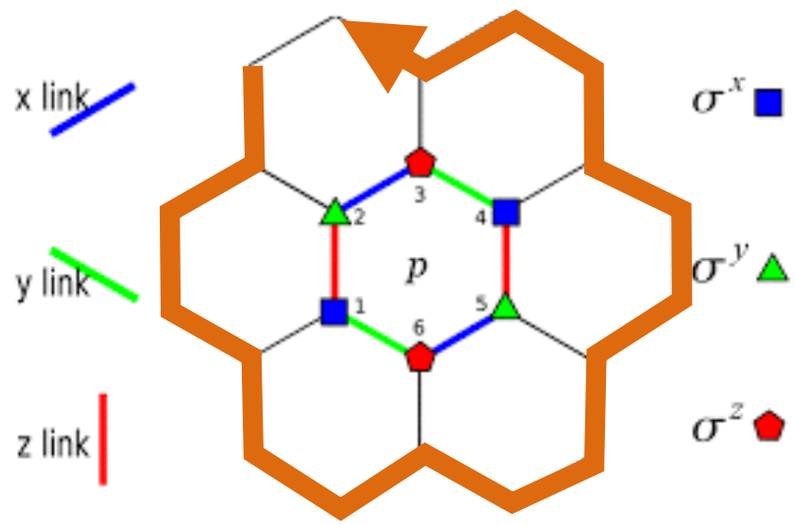

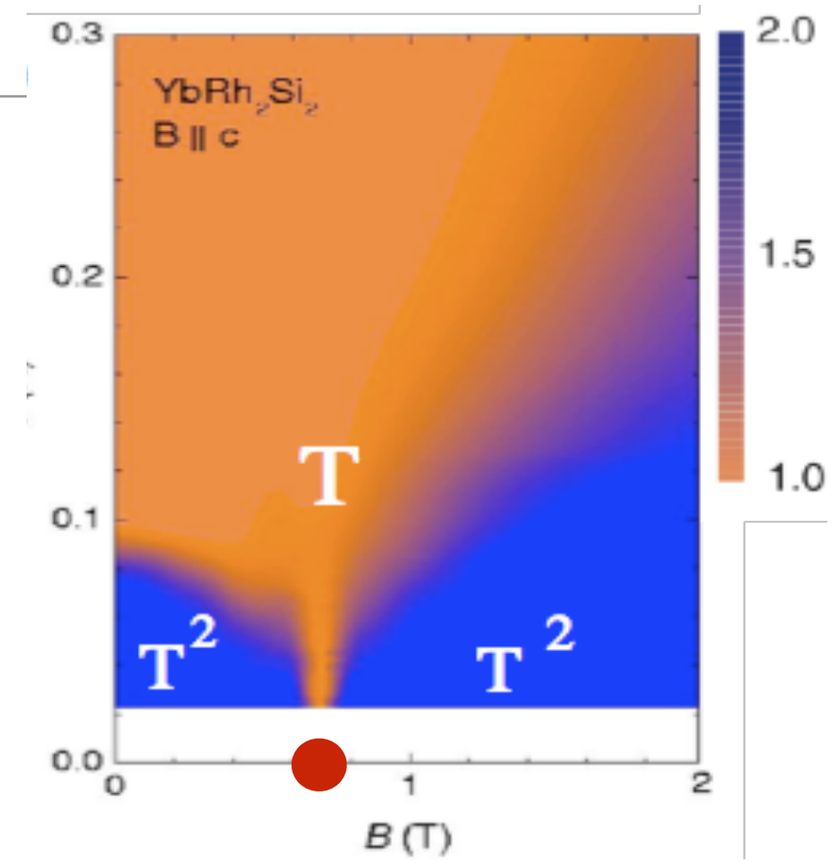

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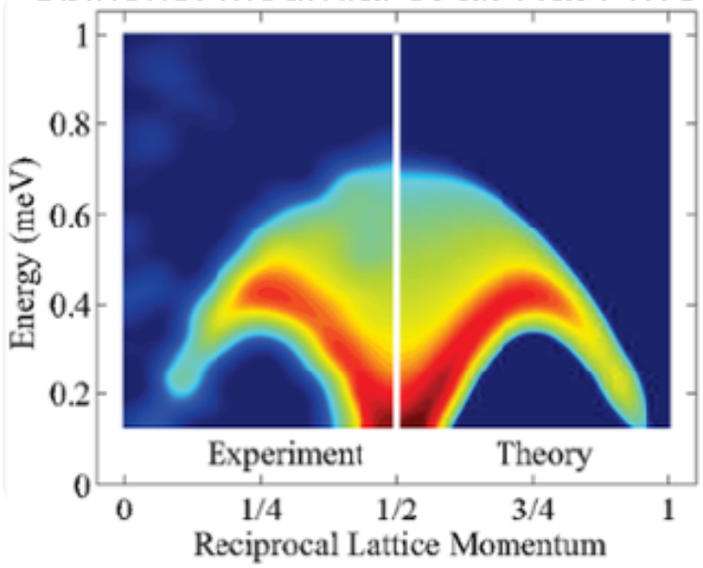

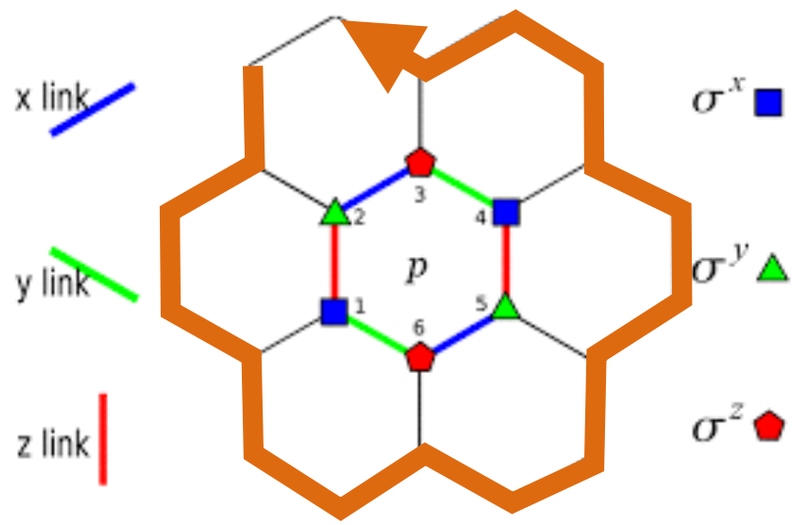

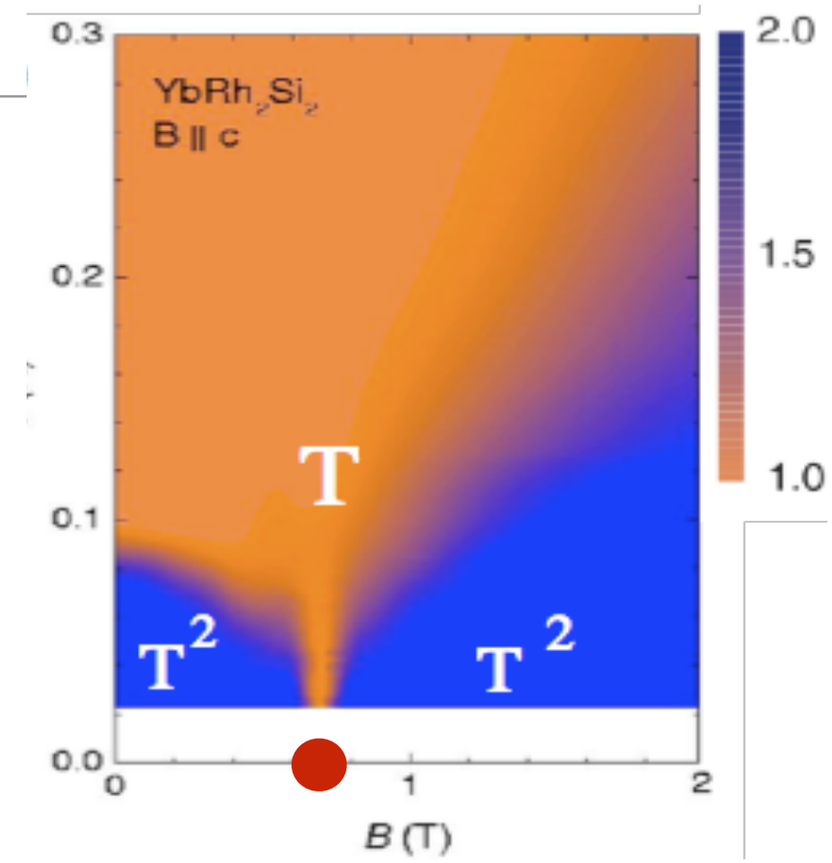

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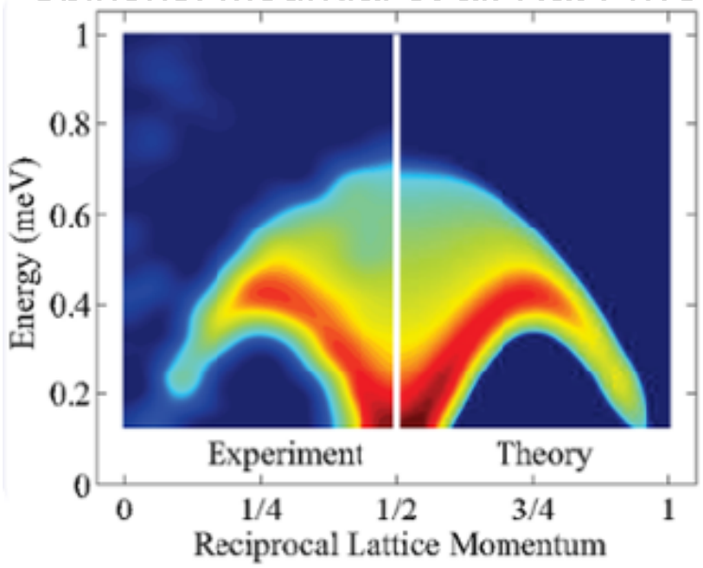

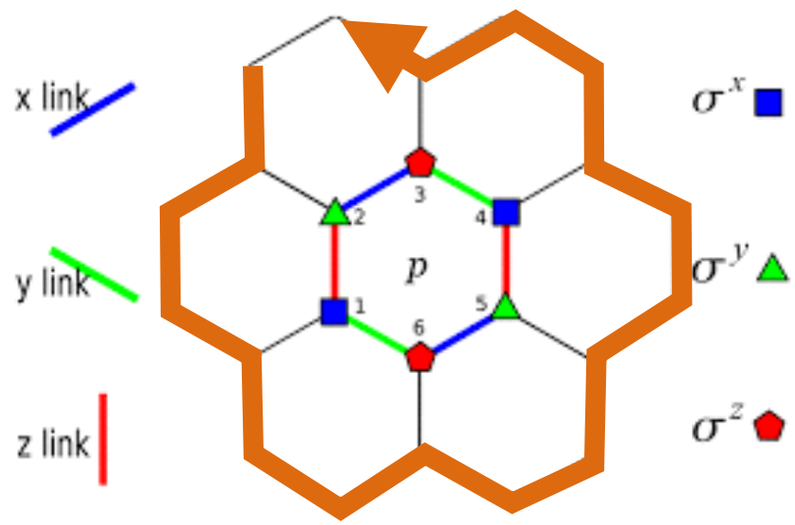

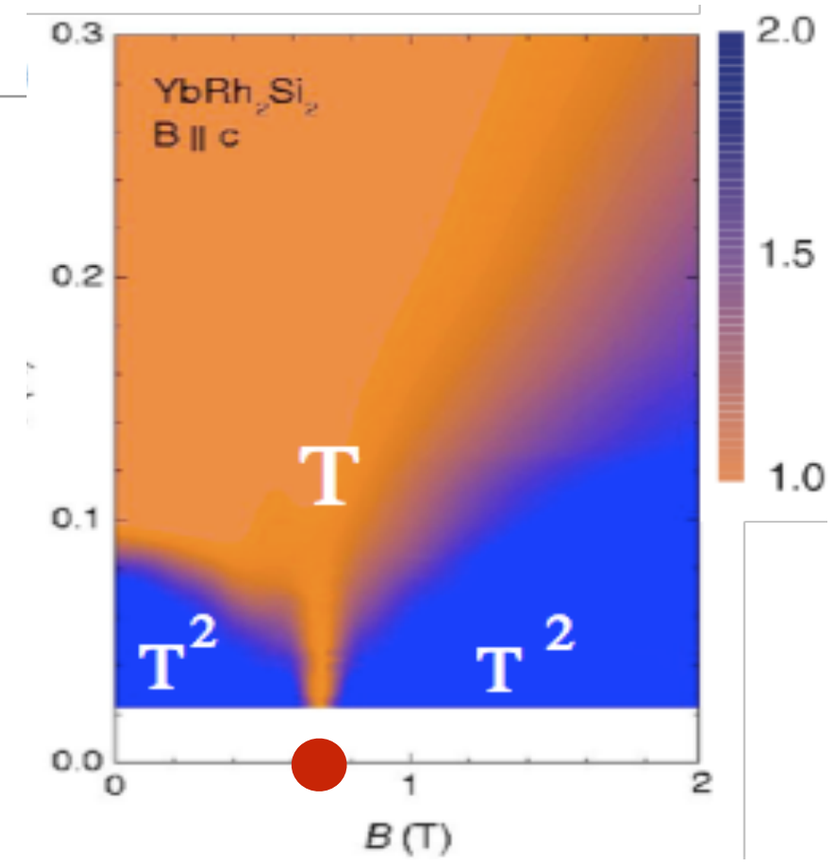

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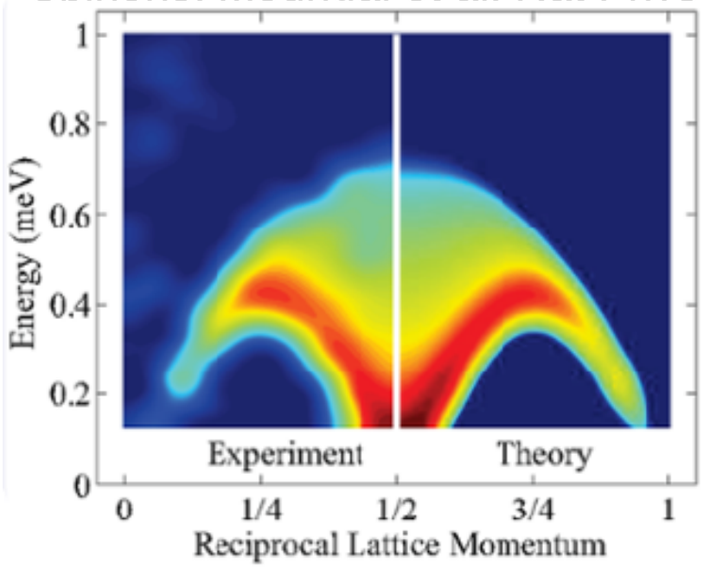

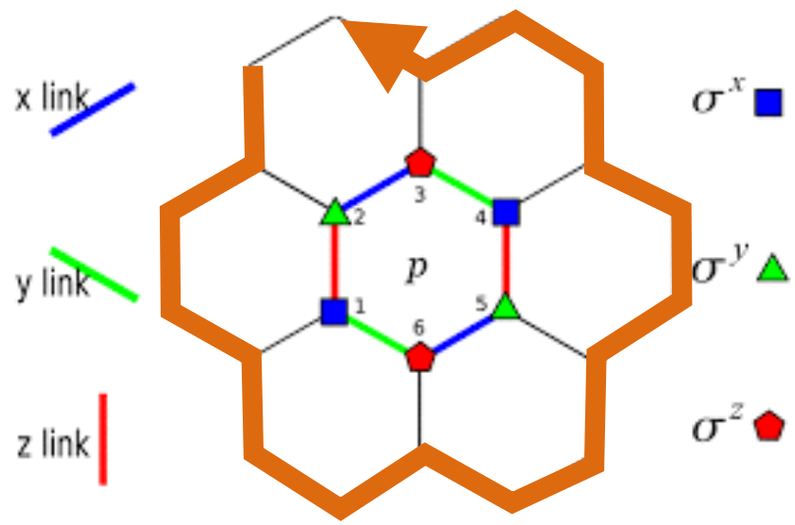

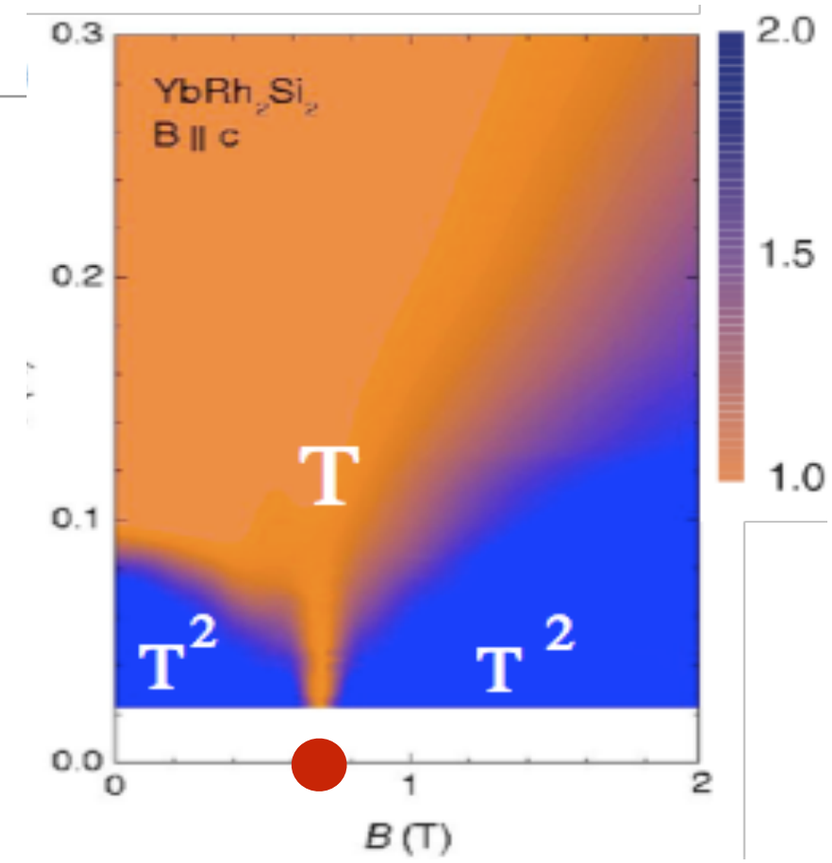

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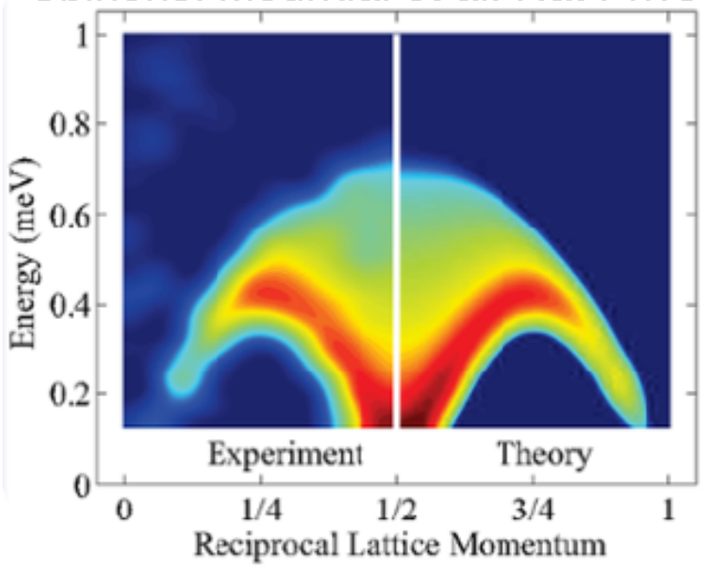

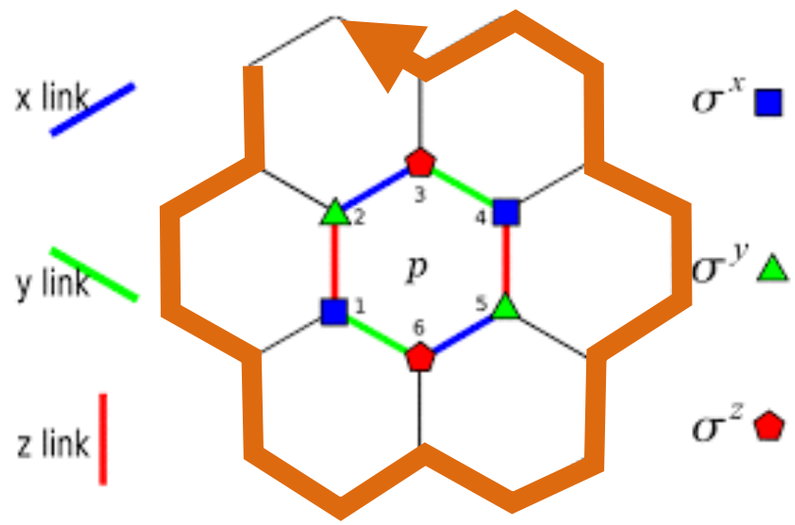

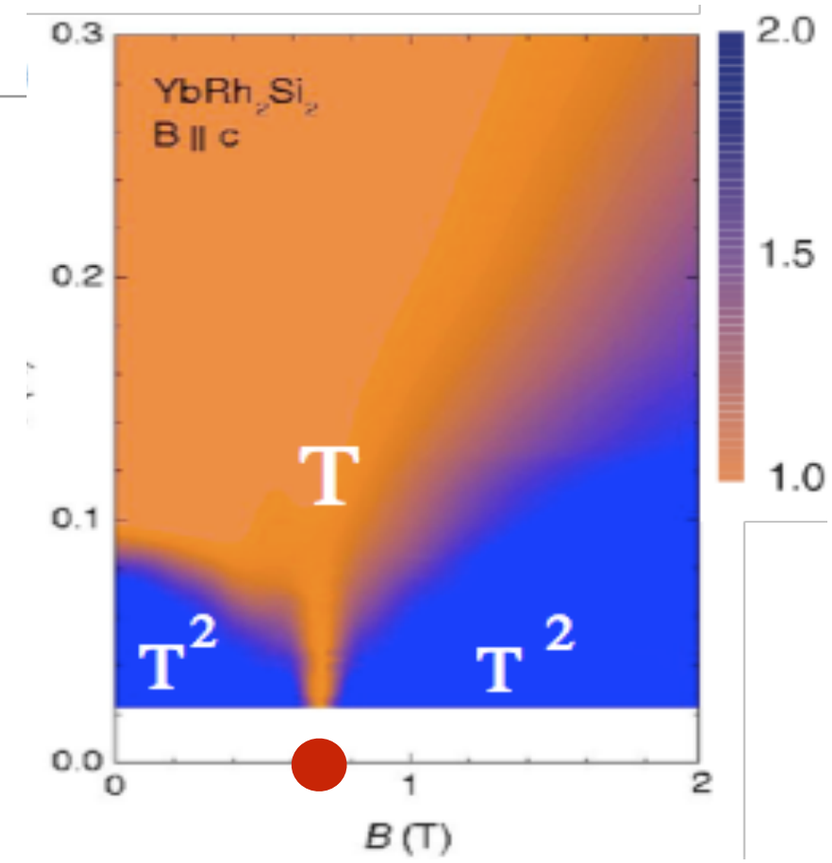

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Conjecture:

Order can fractionalize

- Onset at Deconfined QCP
- ODRLO into the time domain
- Half integer OPS are possible.

Various Motivations

Motivation: Kondo Lattice Physics

$$H = -t \sum_{(i,j)\sigma} (c^{\dagger}{}_{i\sigma} c_{j\sigma} + \mathrm{H.c}) + J \sum_{j,\alpha\beta} \vec{\sigma}_j \cdot \vec{S}_j,$$

"Platonic" Kondo (Andy Millis)

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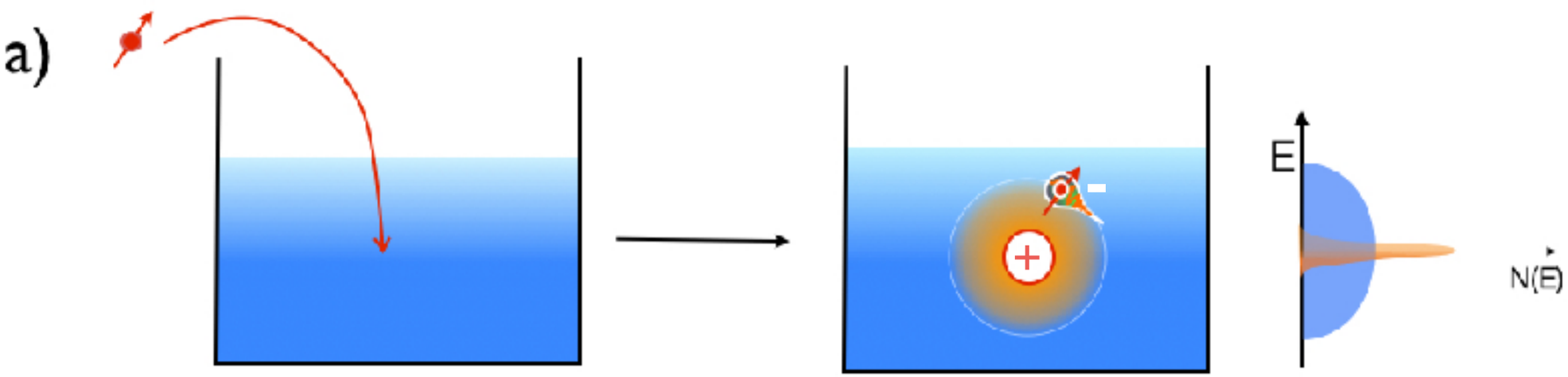

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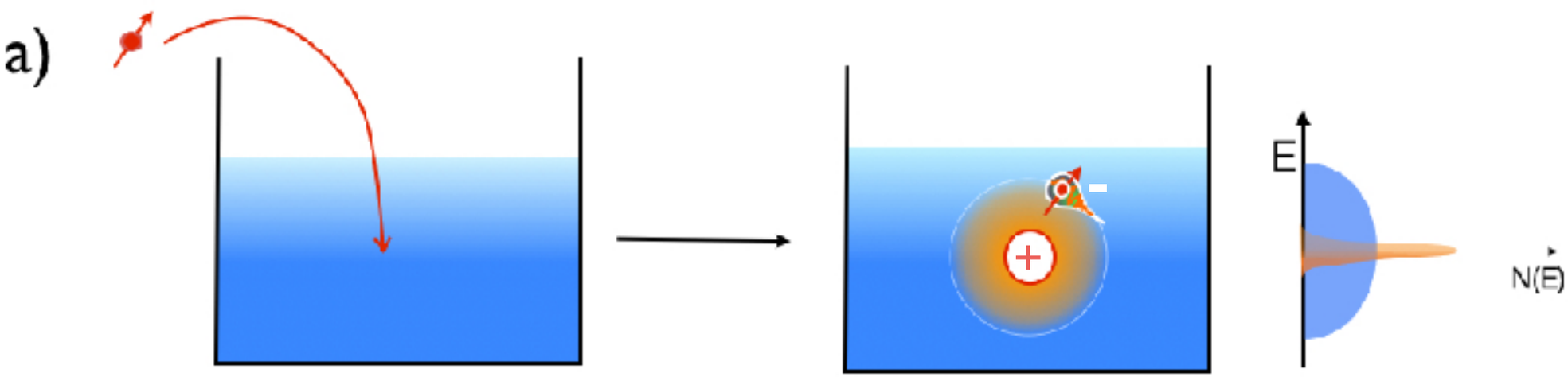

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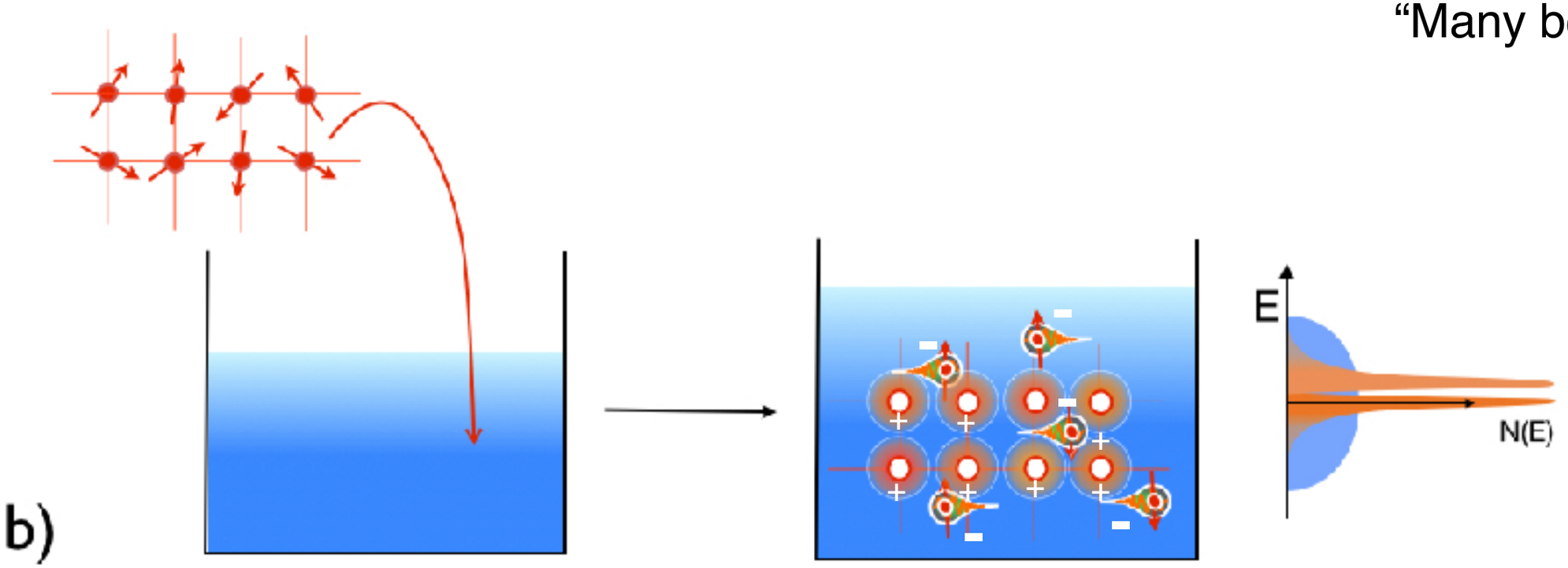

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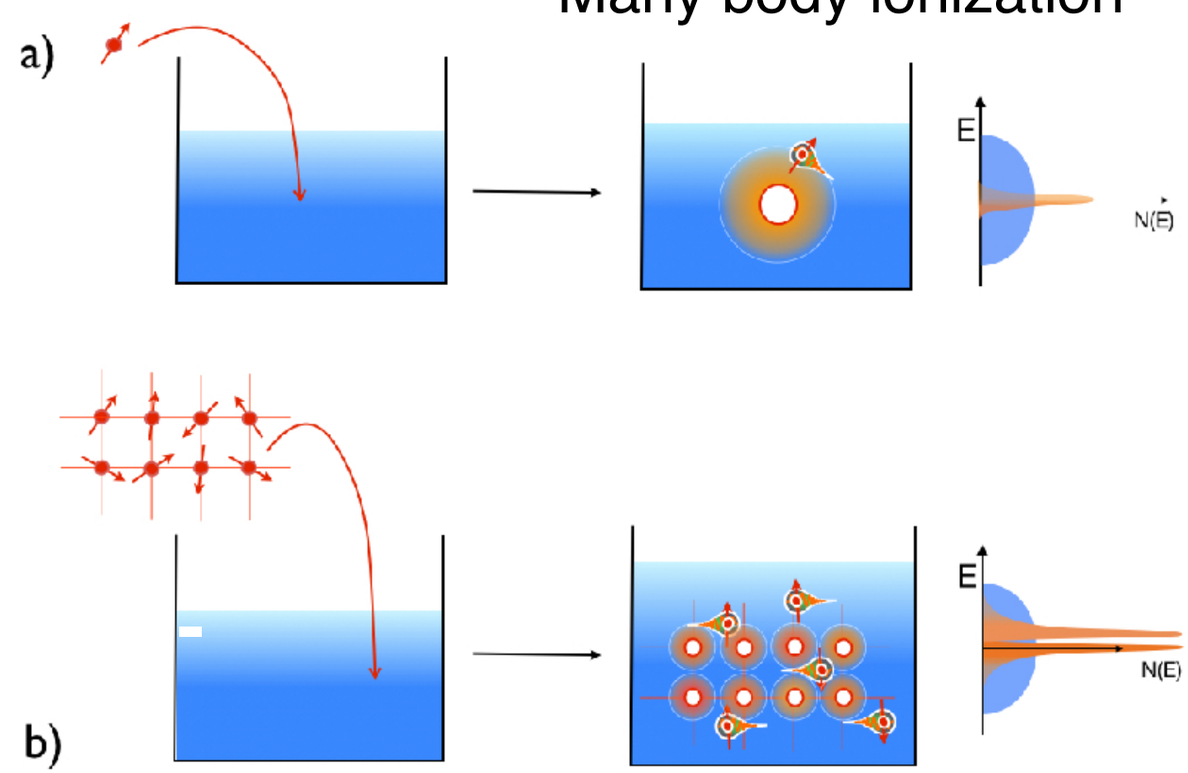

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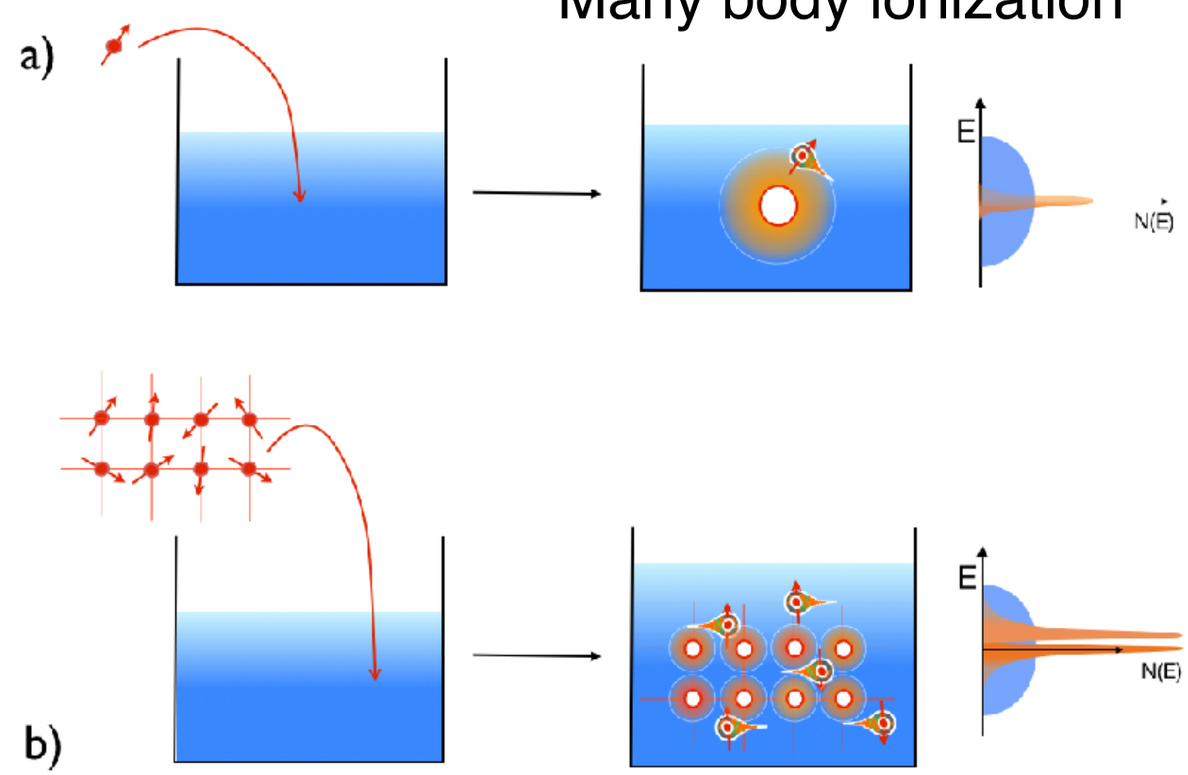

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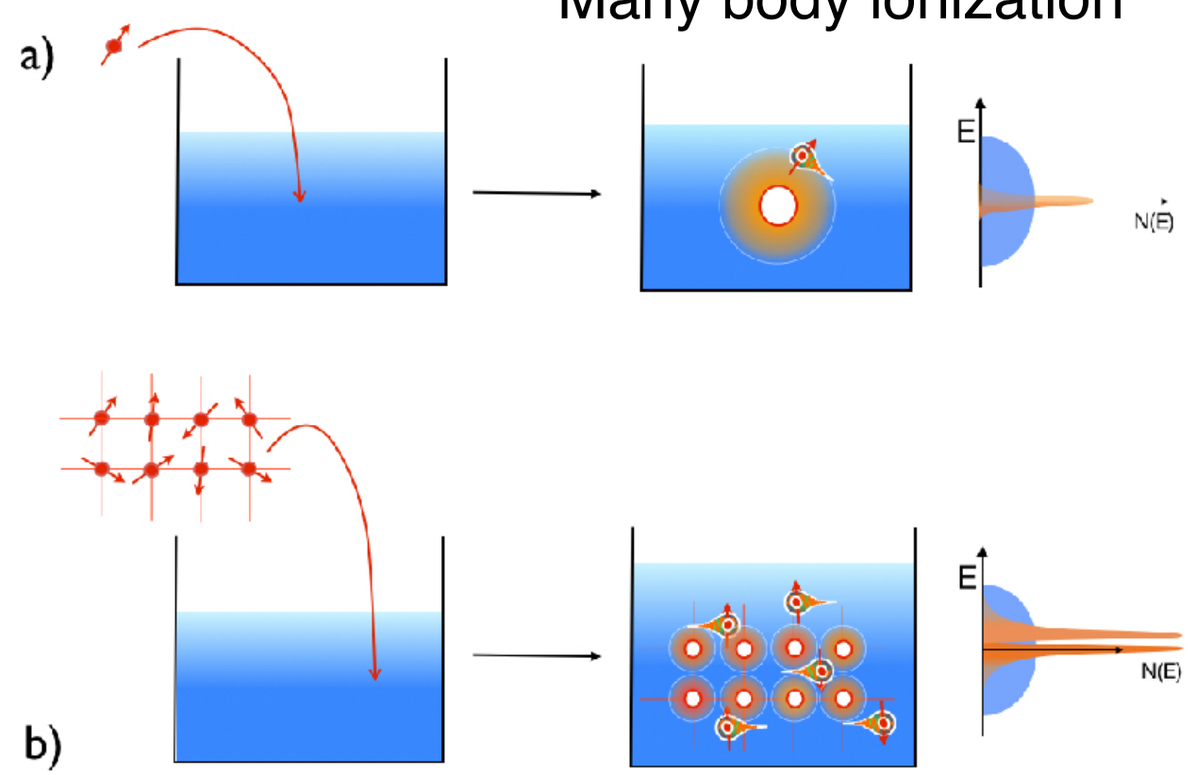

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$\vec{S} \rightarrow f^{\dagger}_{\alpha} f_{\beta}$ Spin *Fractionalization*

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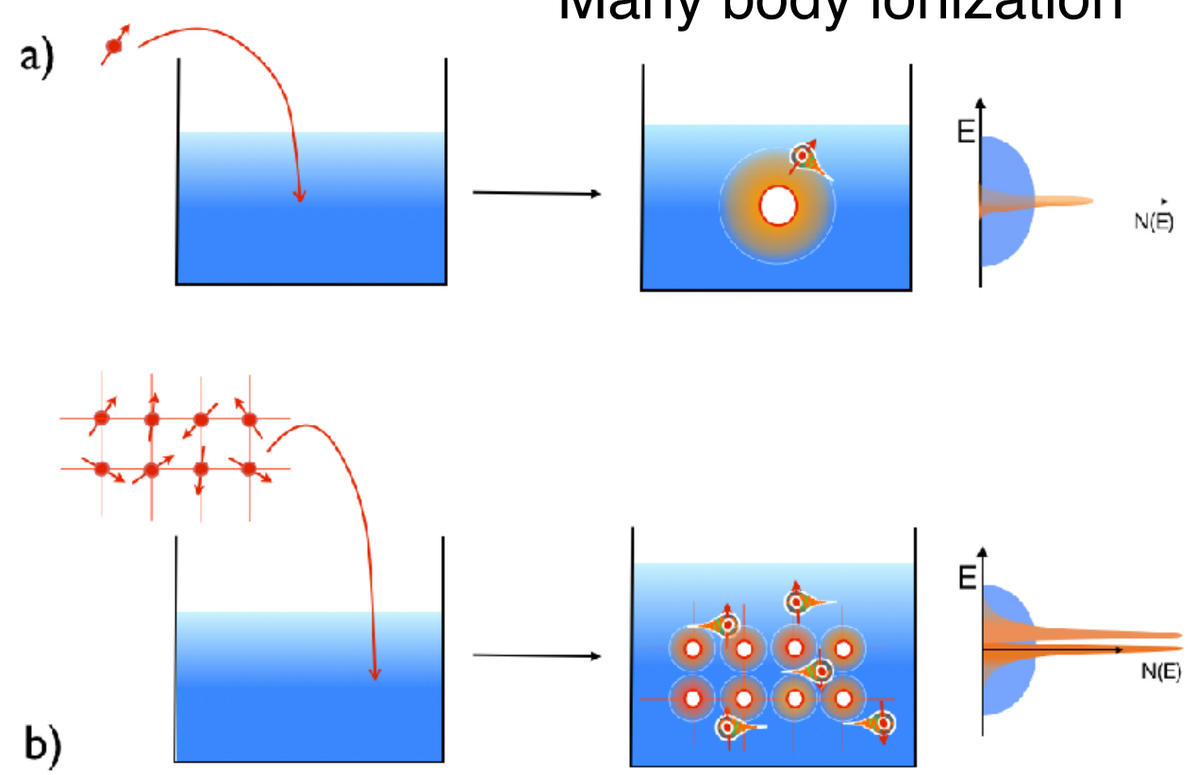

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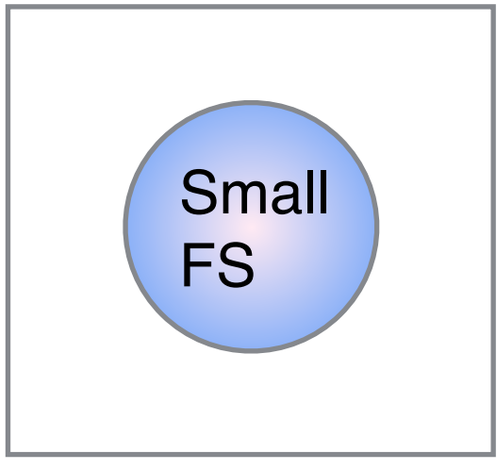

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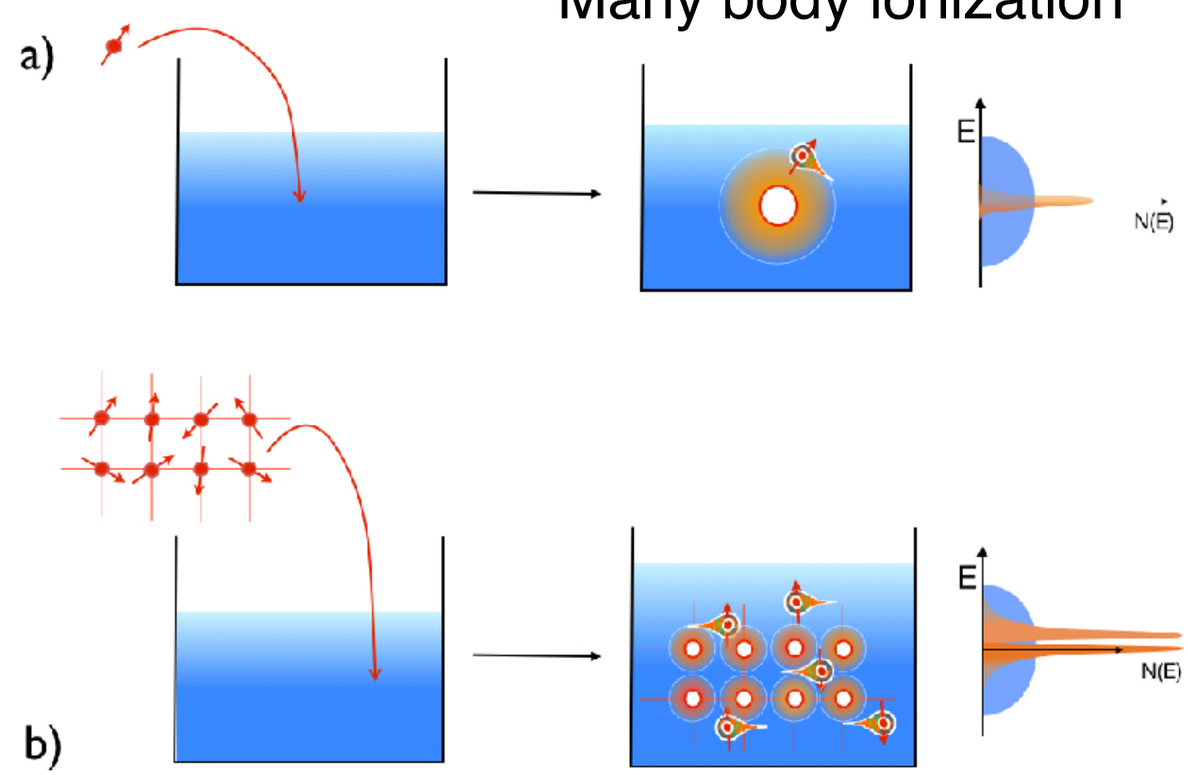

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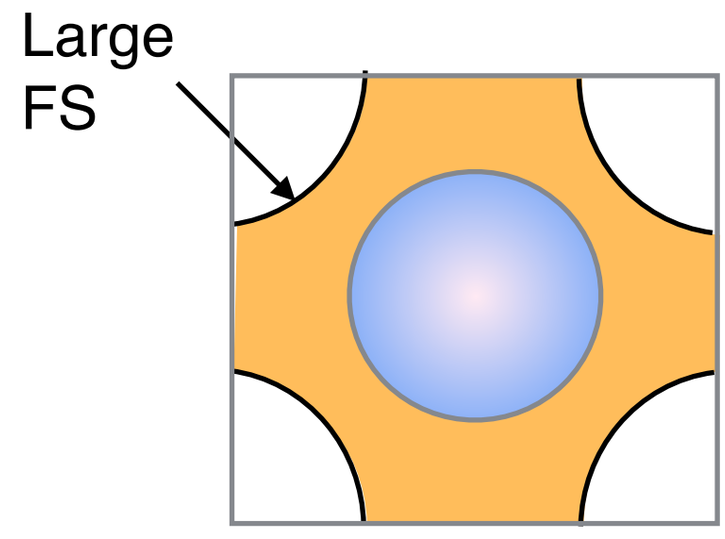

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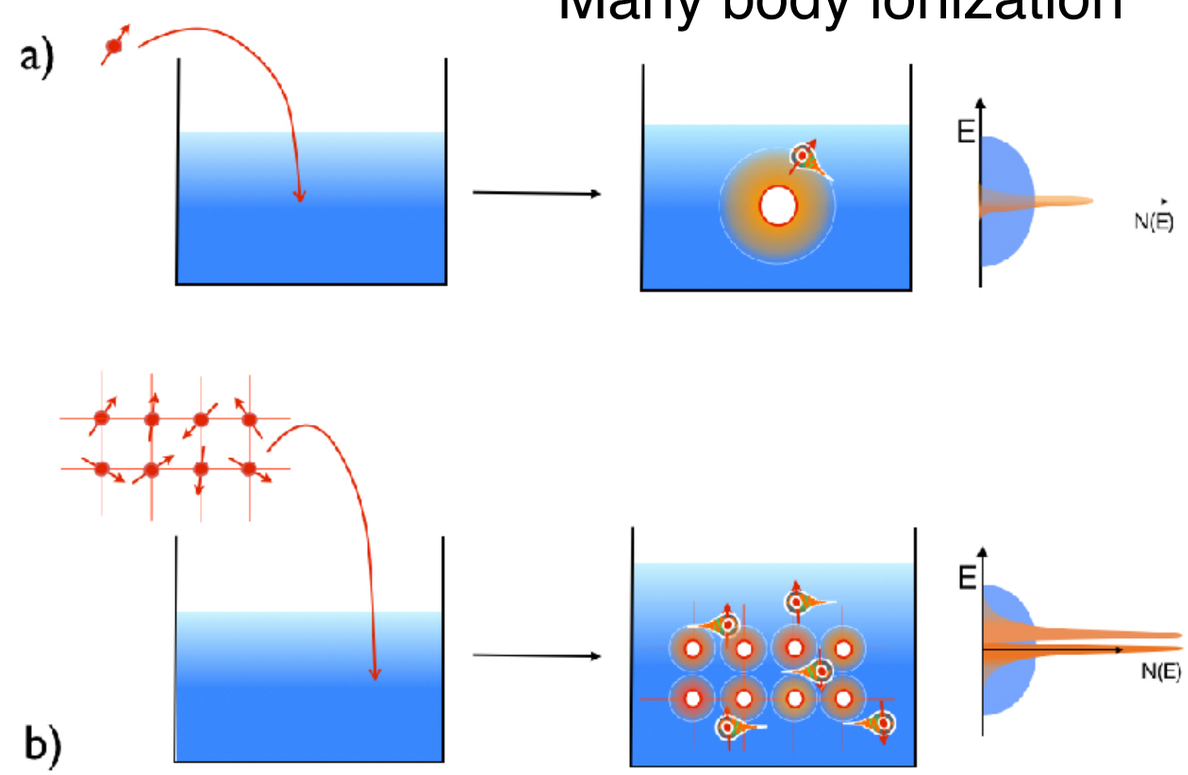

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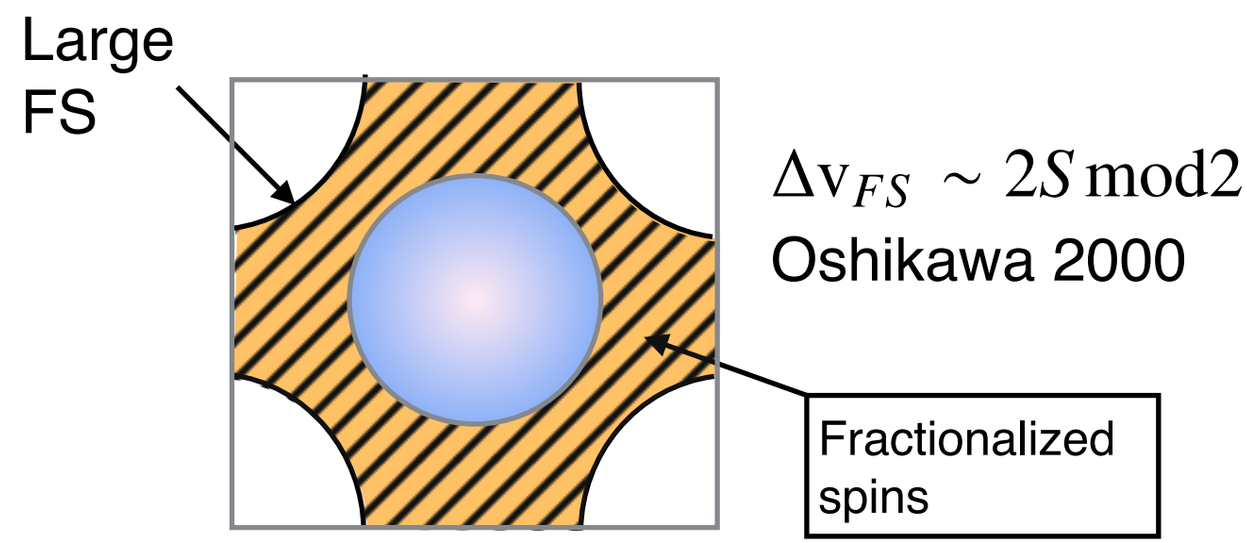

$\Delta v_{FS} \sim 2S \bmod 2$

Oshikawa 2000

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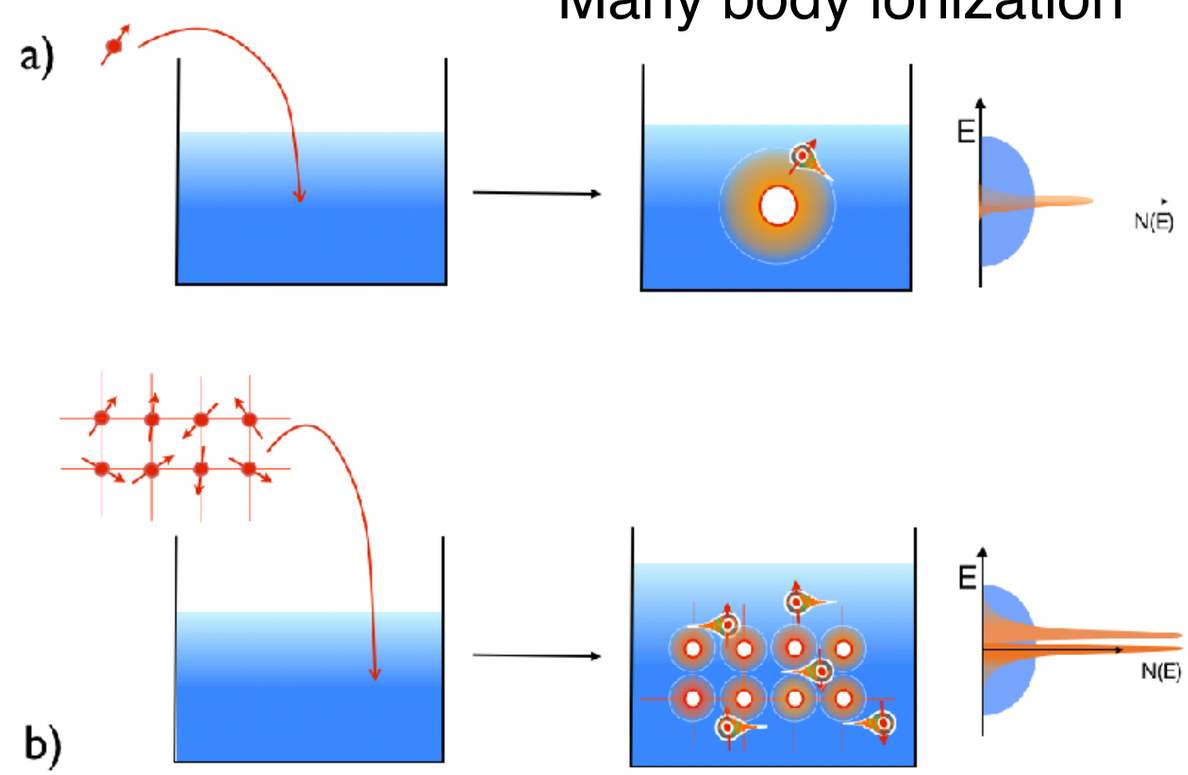

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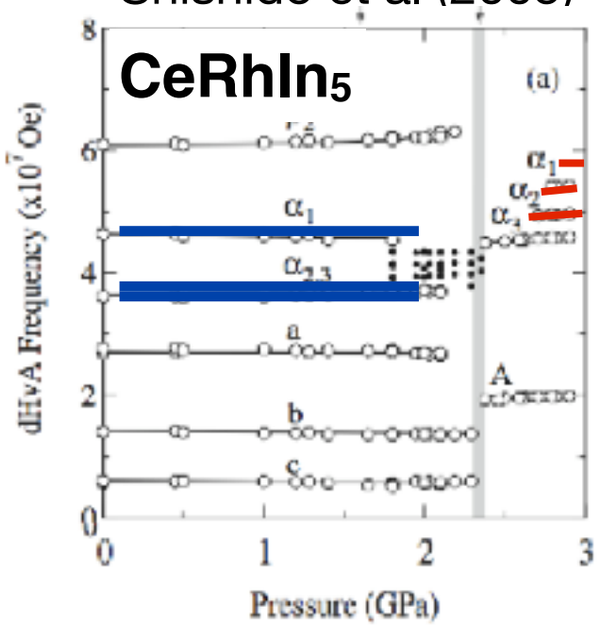

Pressure driven Fractionalization

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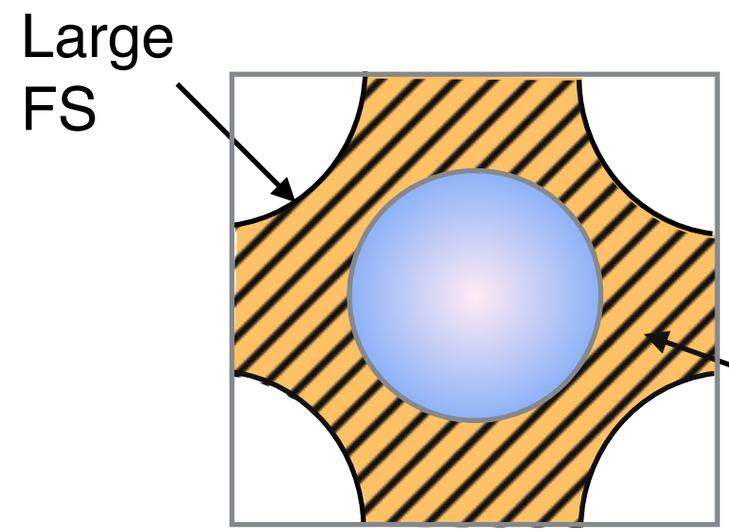

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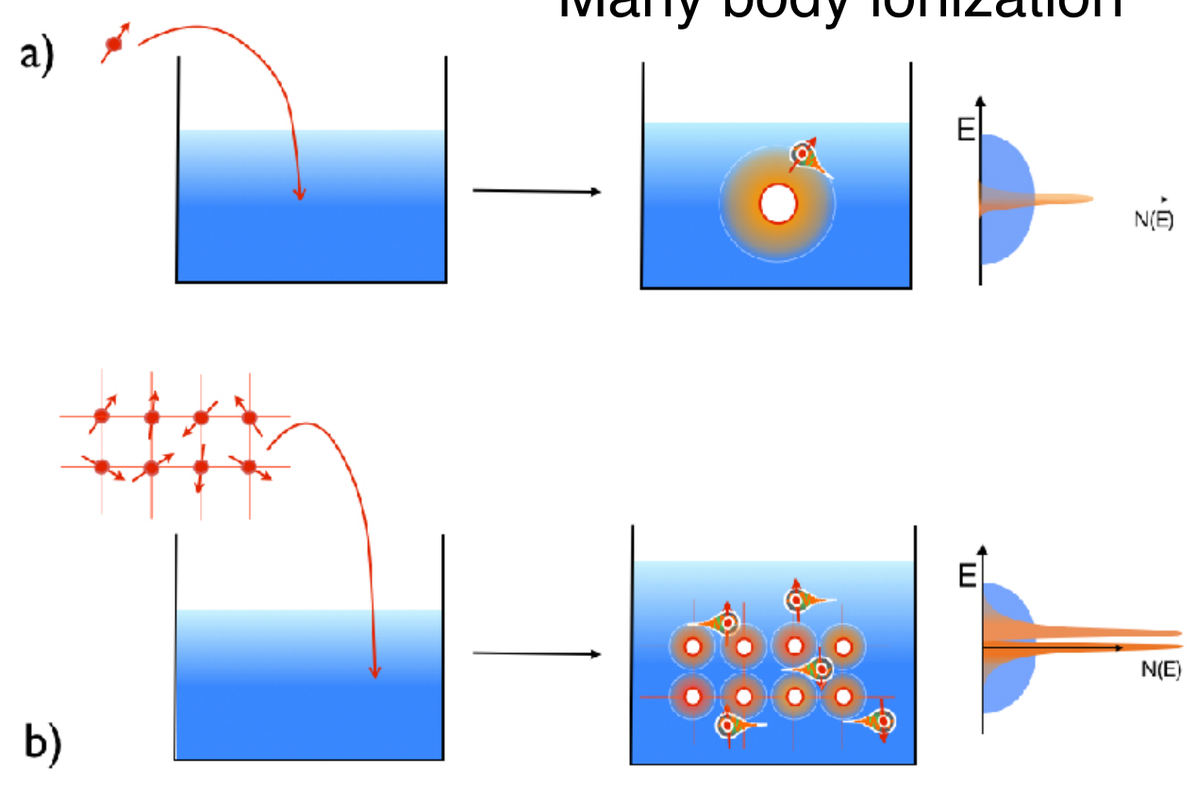

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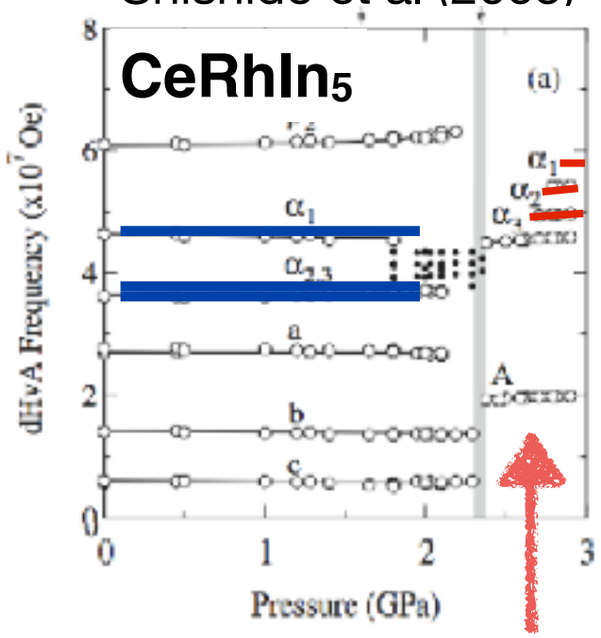

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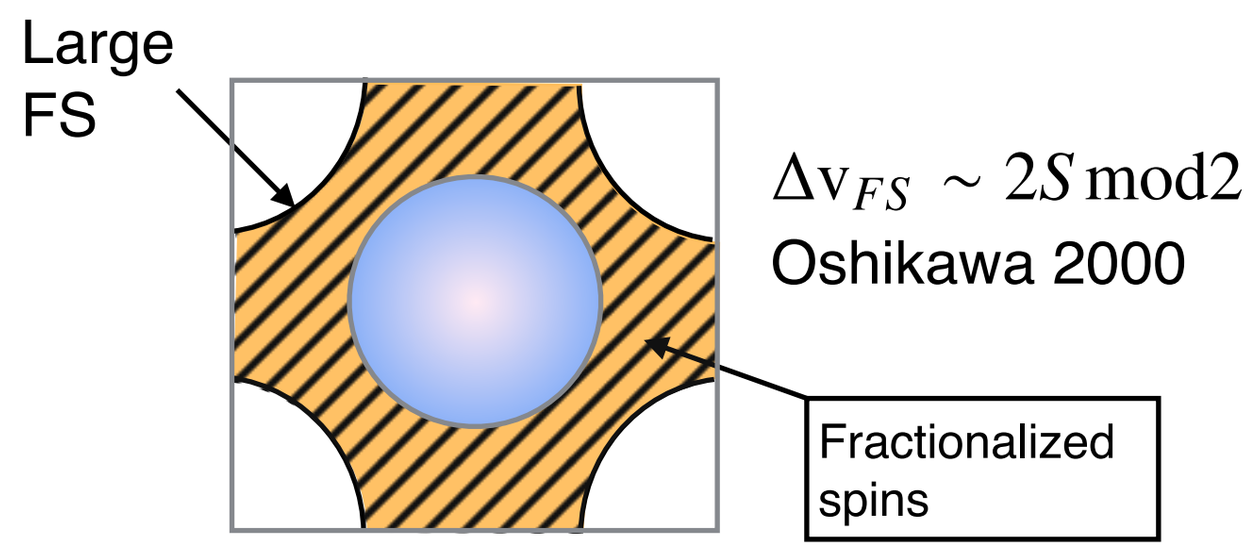

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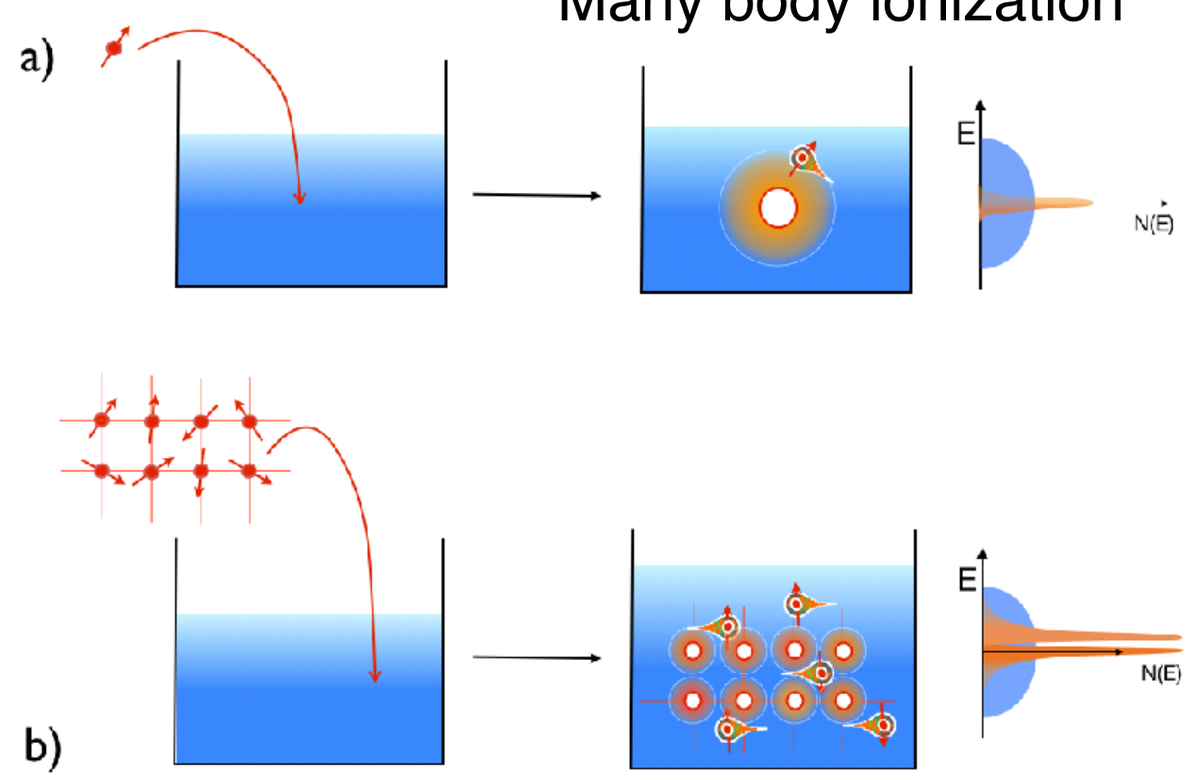
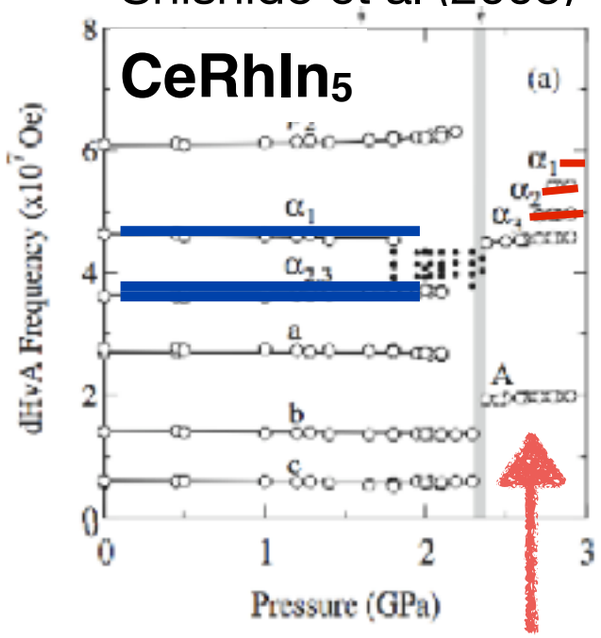
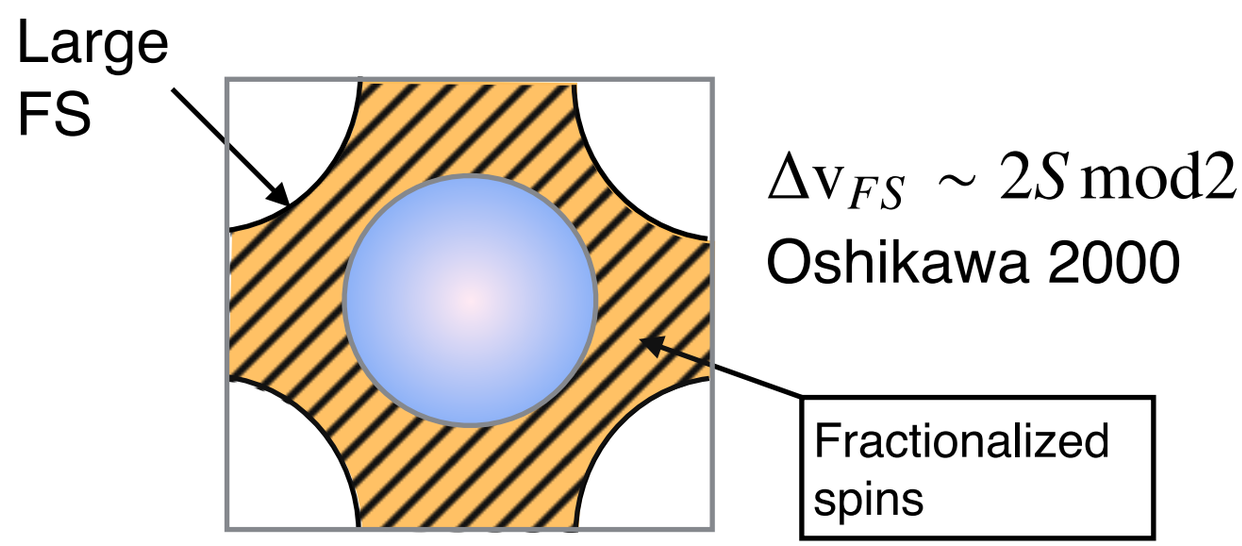


Motivation: Kondo Lattice Physics

Shishido et al (2005)



Pressure driven Fractionalization



$\Delta v_{FS} \sim 2S \bmod 2$

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But viewed from the perspective of the Kondo model, the f-electron is an emergent fractionalization of the spin, as a charged Dirac particle.

$\vec{S} \rightarrow f_\alpha^\dagger f_\beta$ Spin *Fractionalization*

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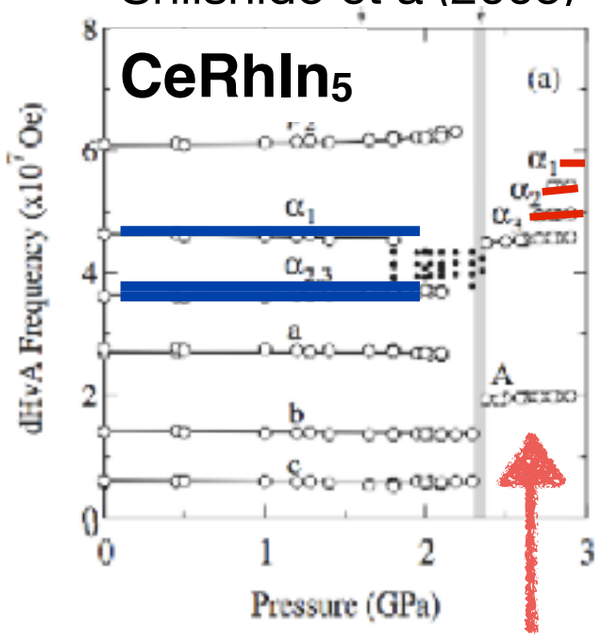

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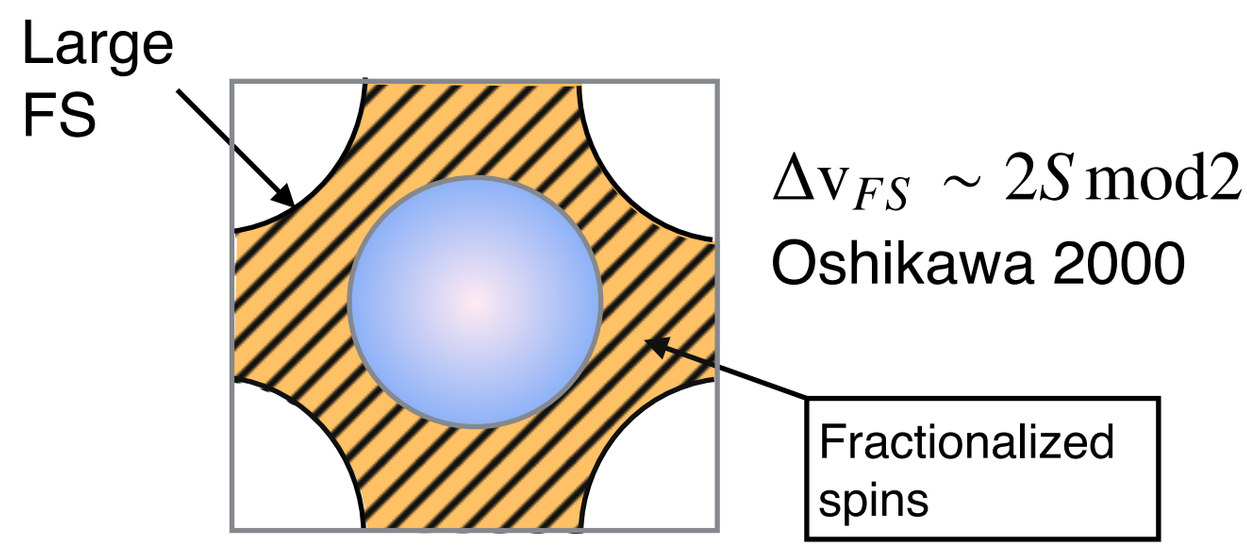

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What other kinds of fractionalization are possible?

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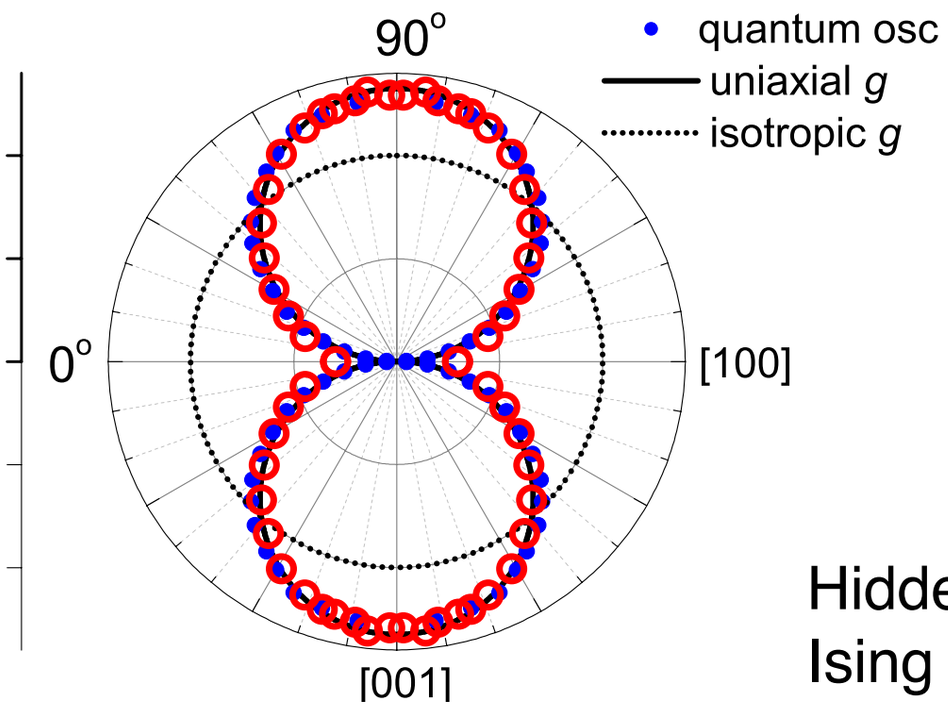

Altarawnah et al, 2011

Hidden Order URu_2Si_2
Ising Quasiparticles.

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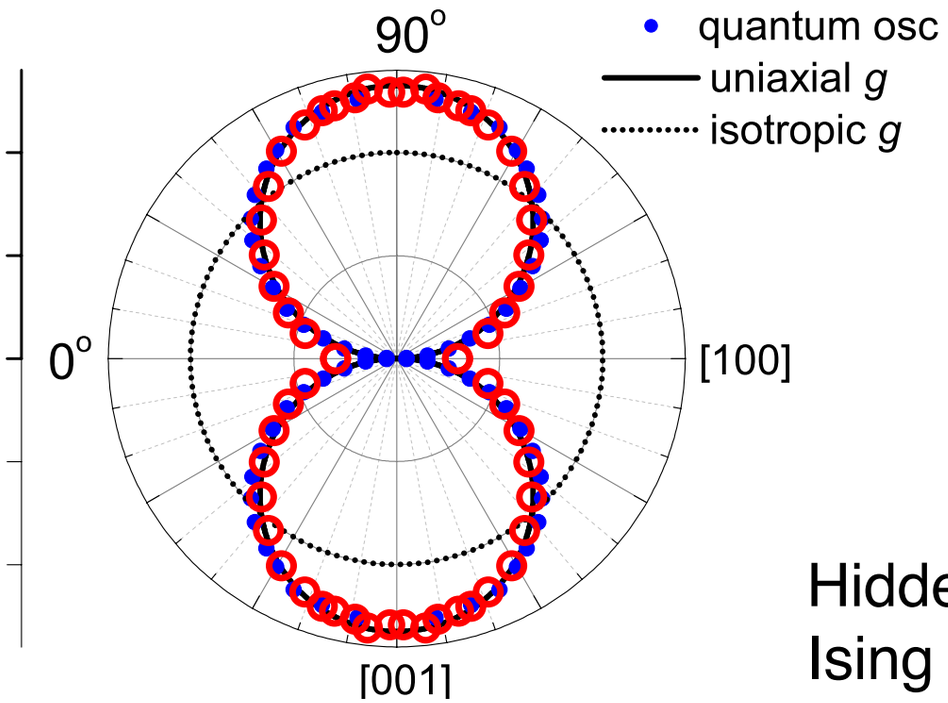

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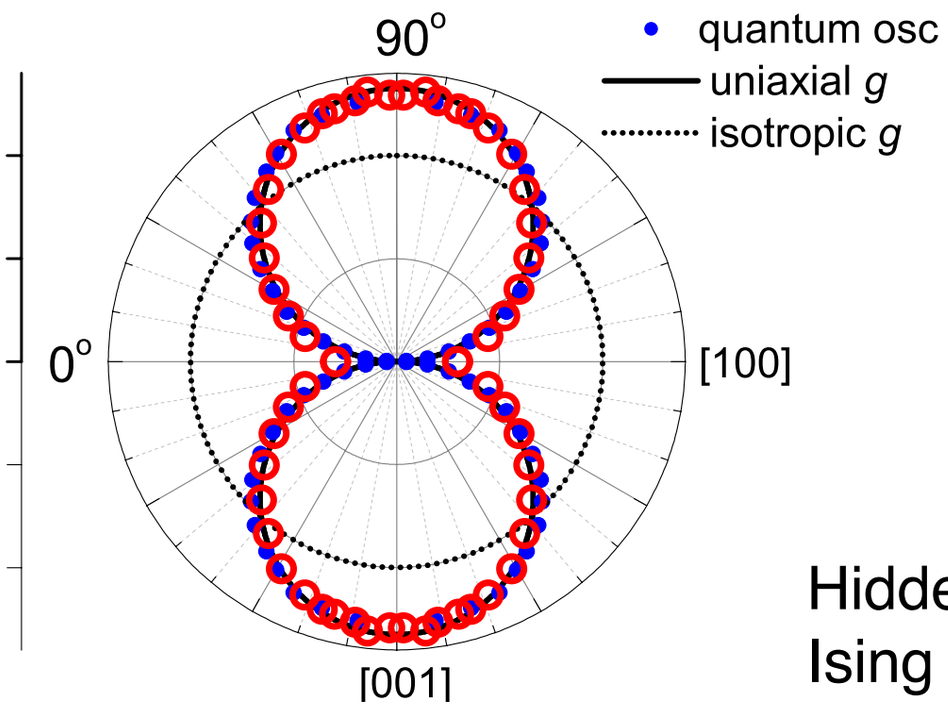

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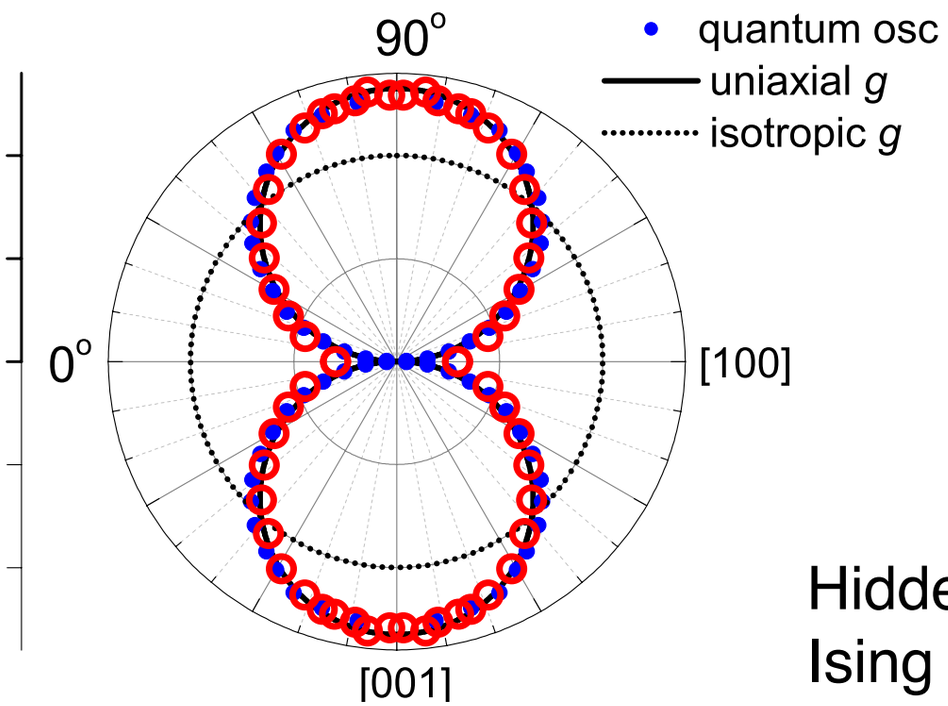

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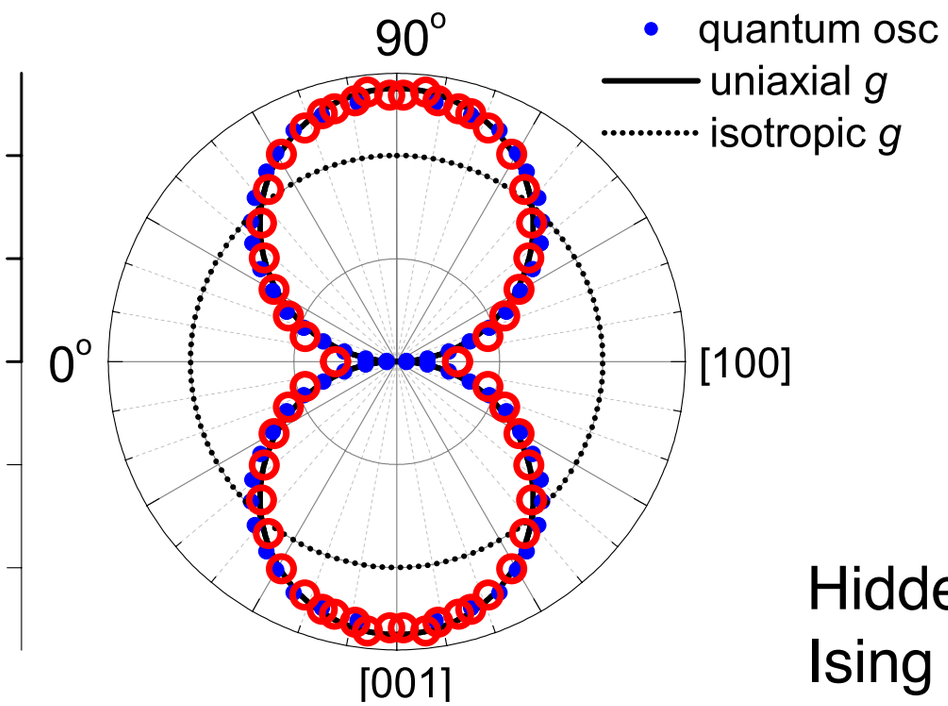

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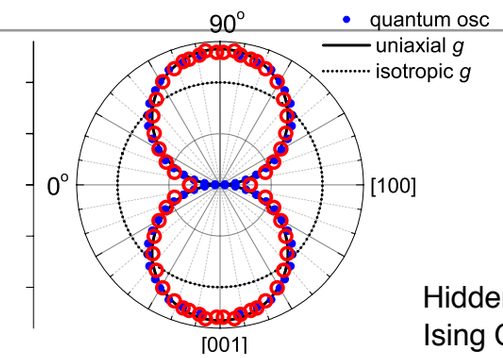

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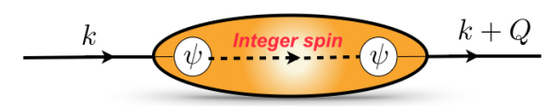

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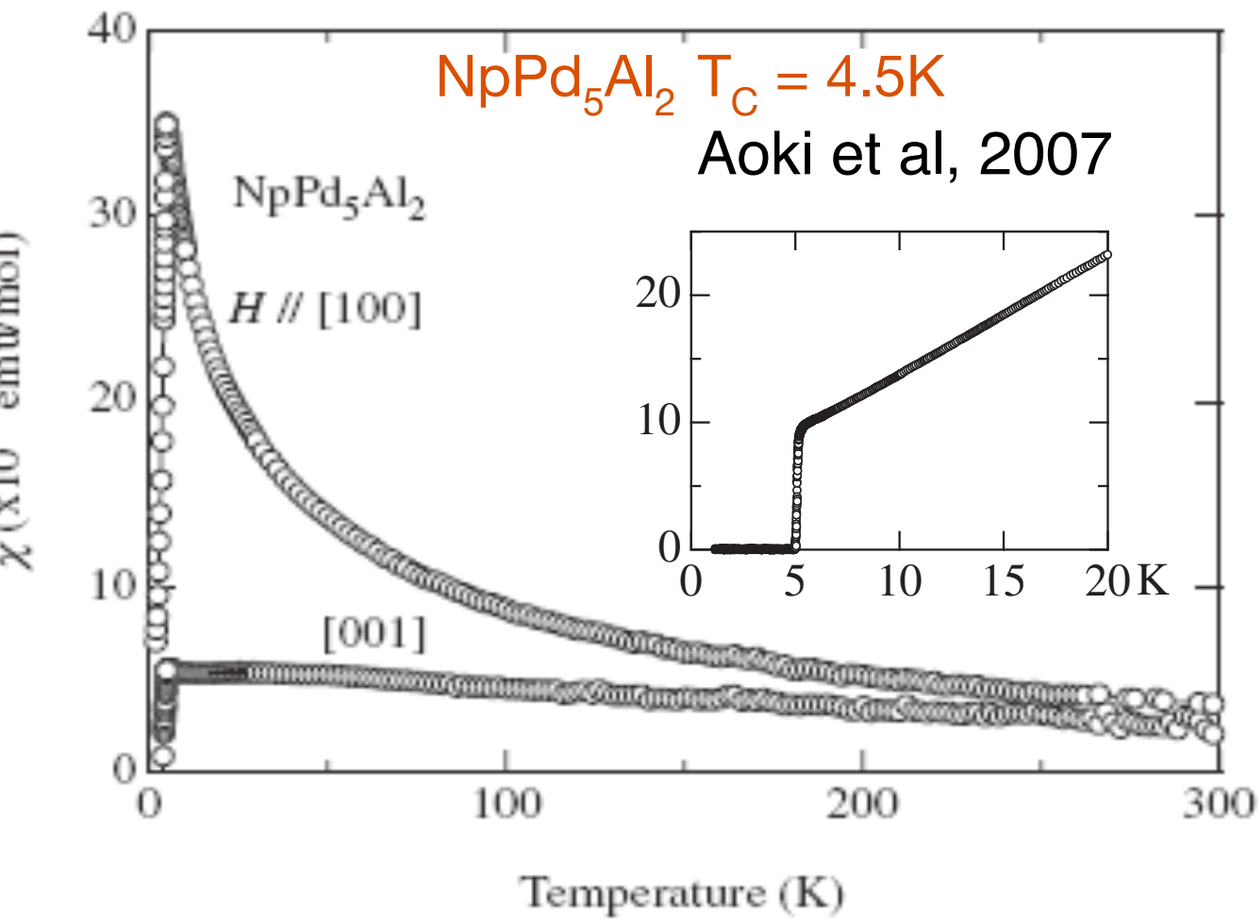
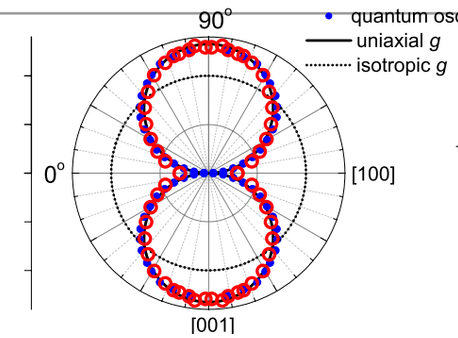


$NpPd_5Al_2$ $T_C = 4.5K$



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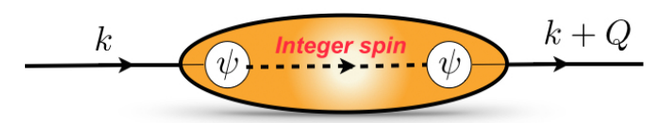

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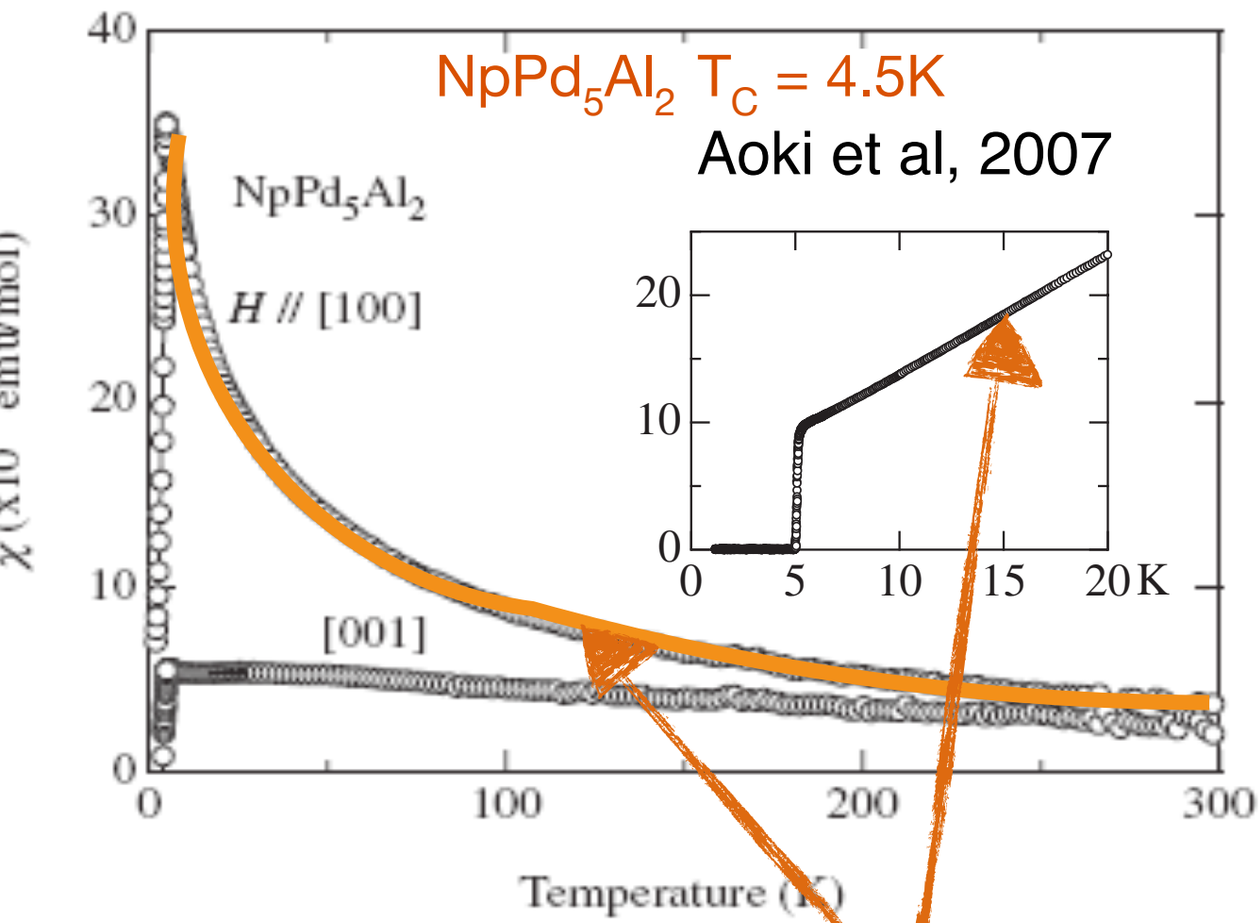

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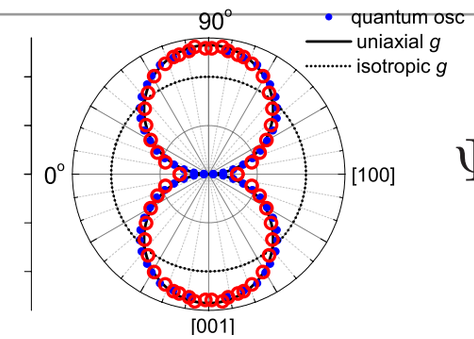

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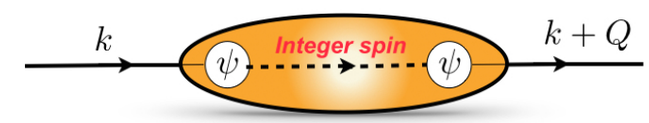

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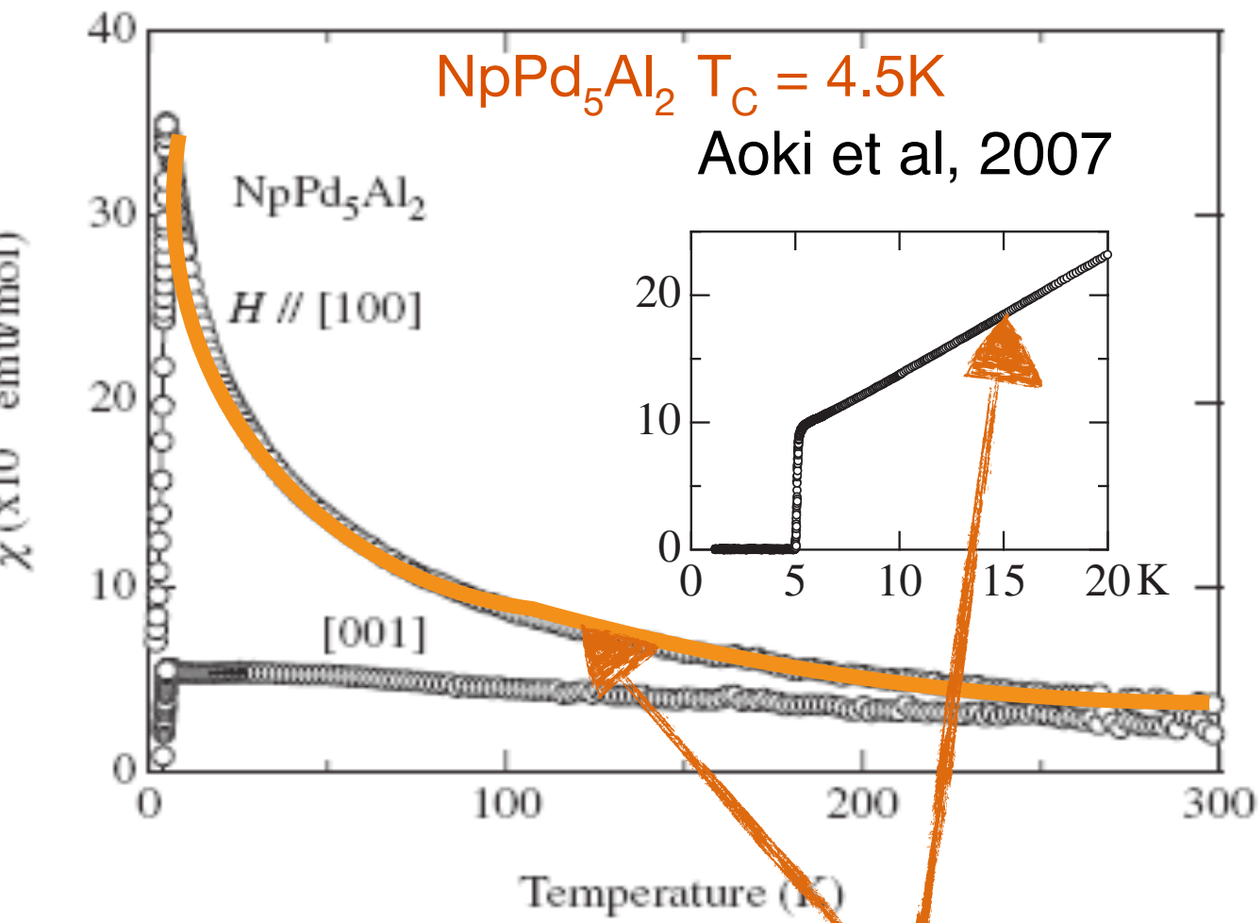

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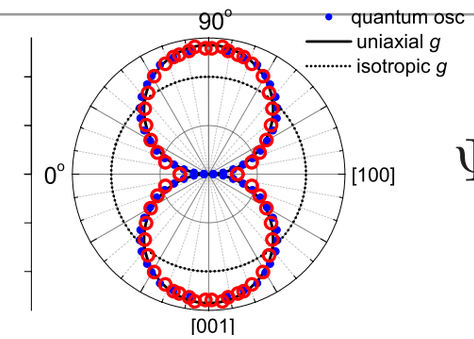

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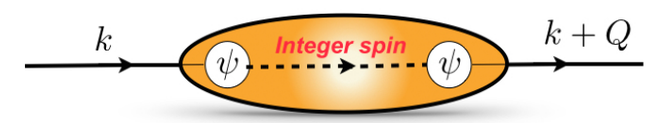

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Here, spin fractionalization appears to coincide with a phase transition.

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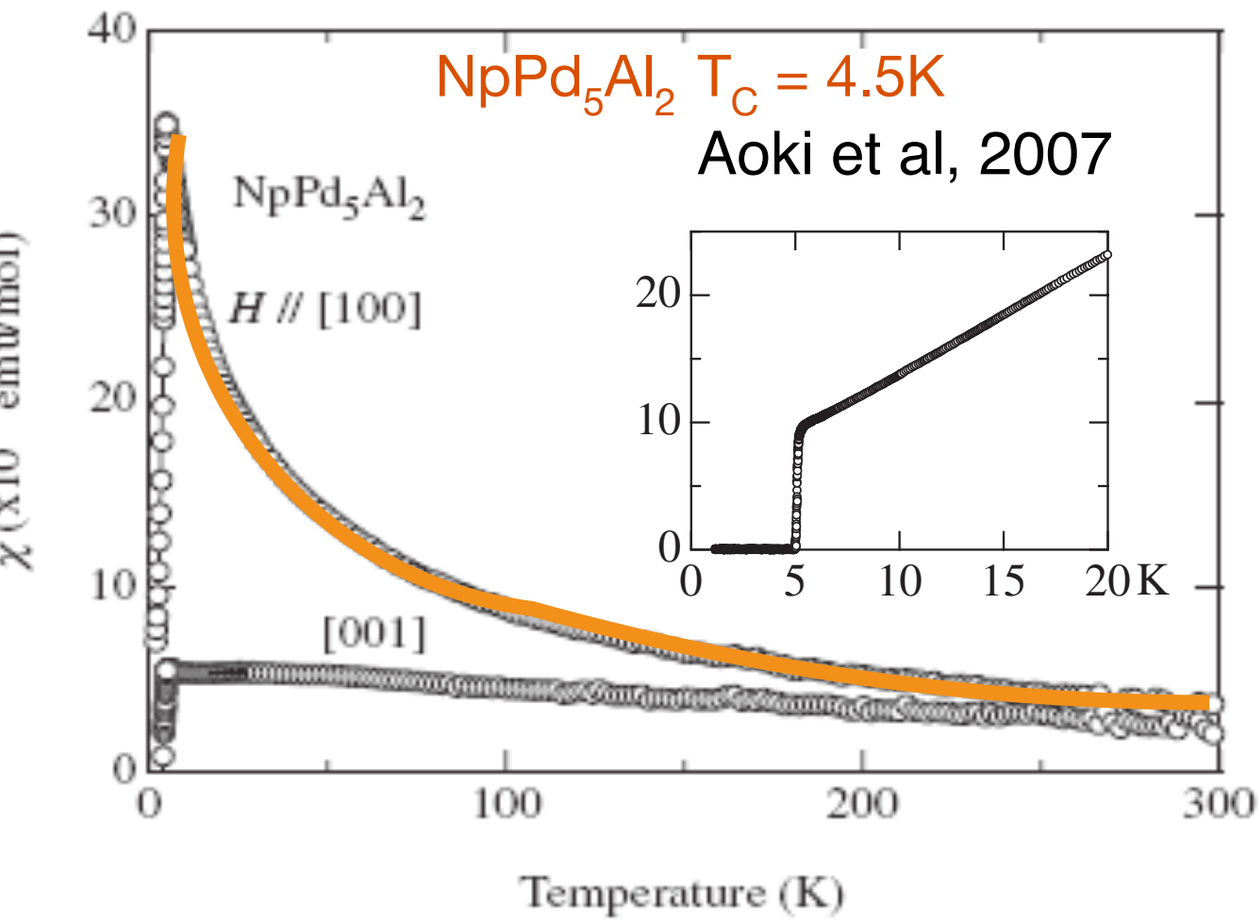

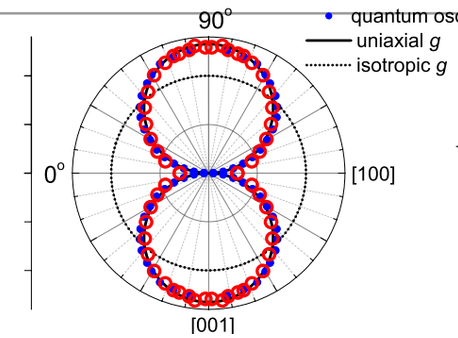

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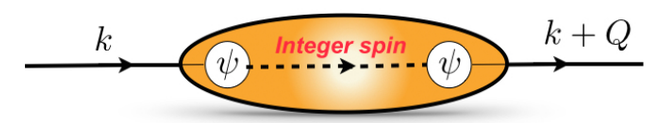

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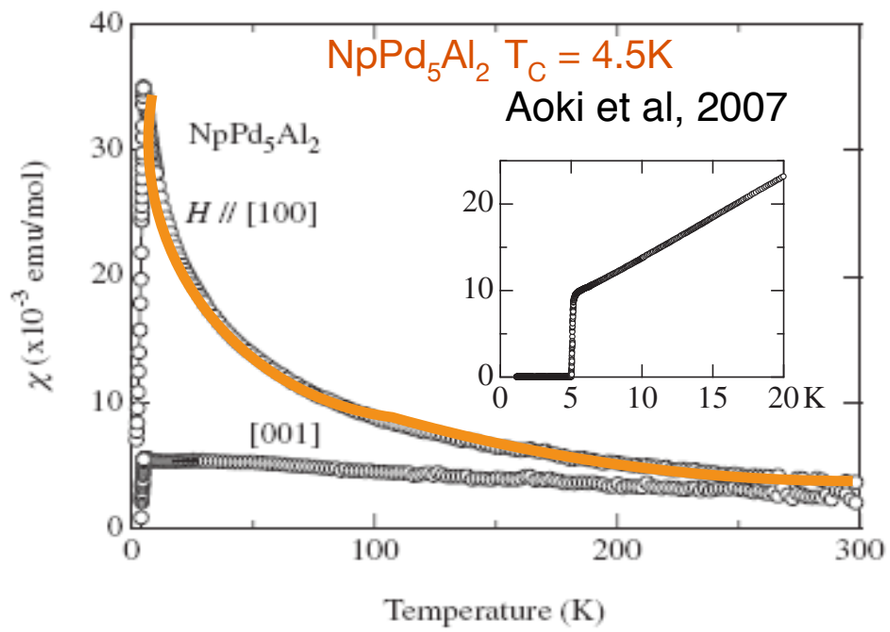
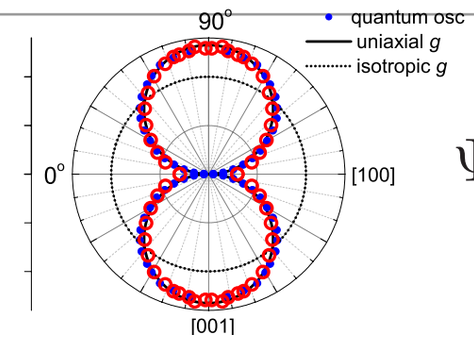


Kondo effect and SC coincide.



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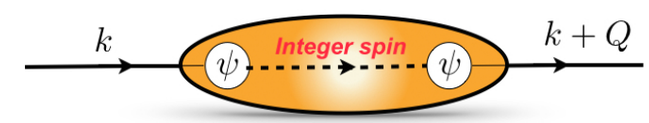

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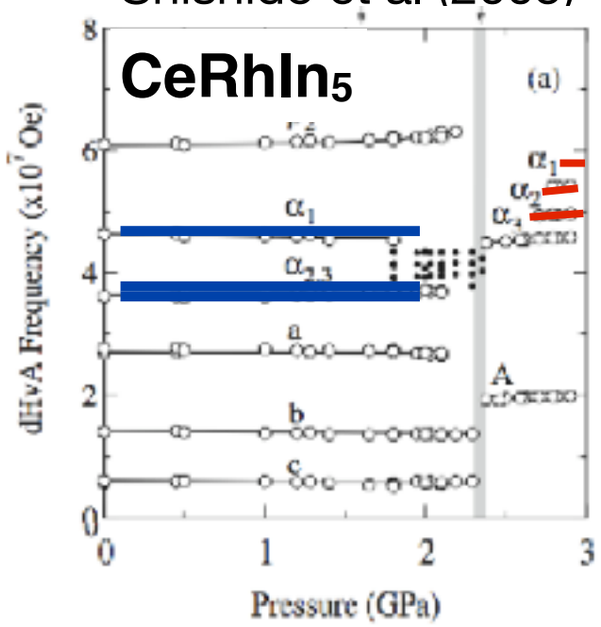

Pressure driven
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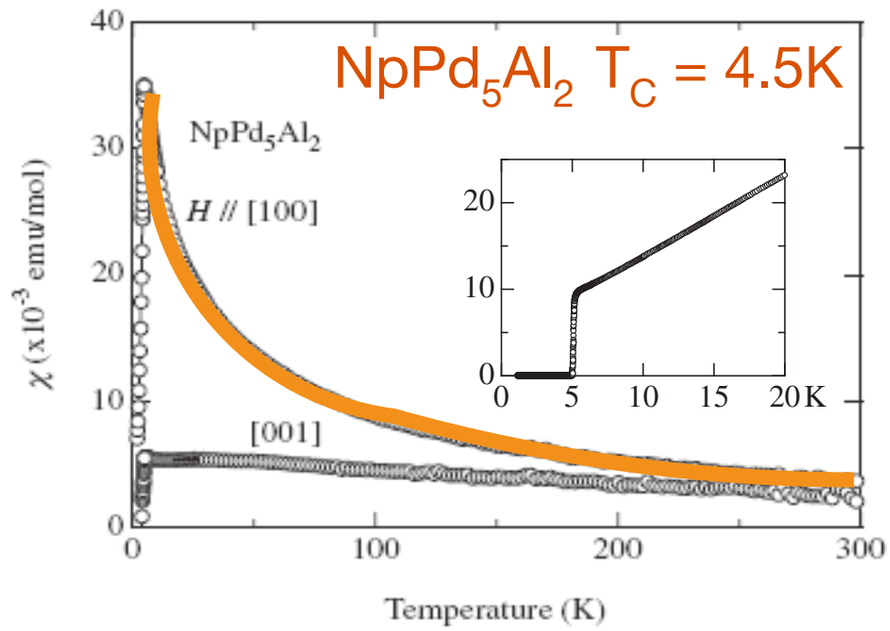

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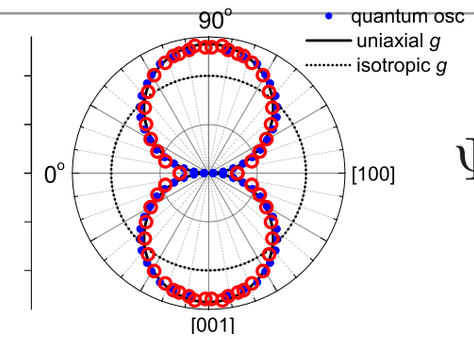

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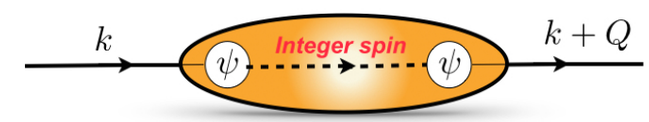

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Hastatic order?

Conjecture:

Order can fractionalize

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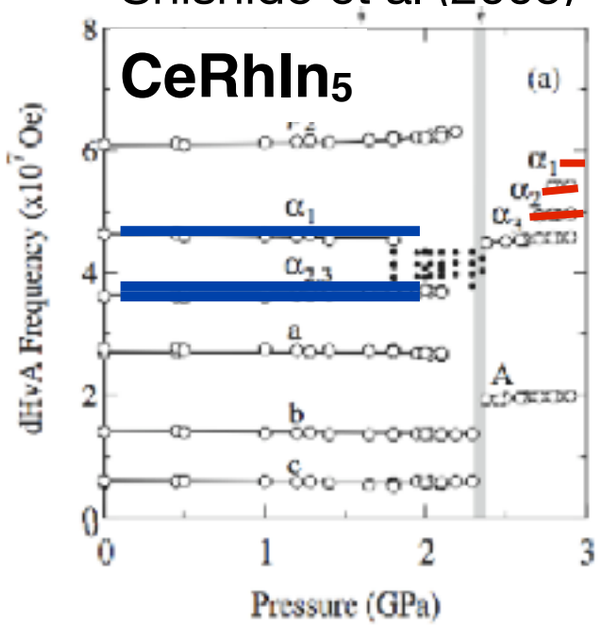

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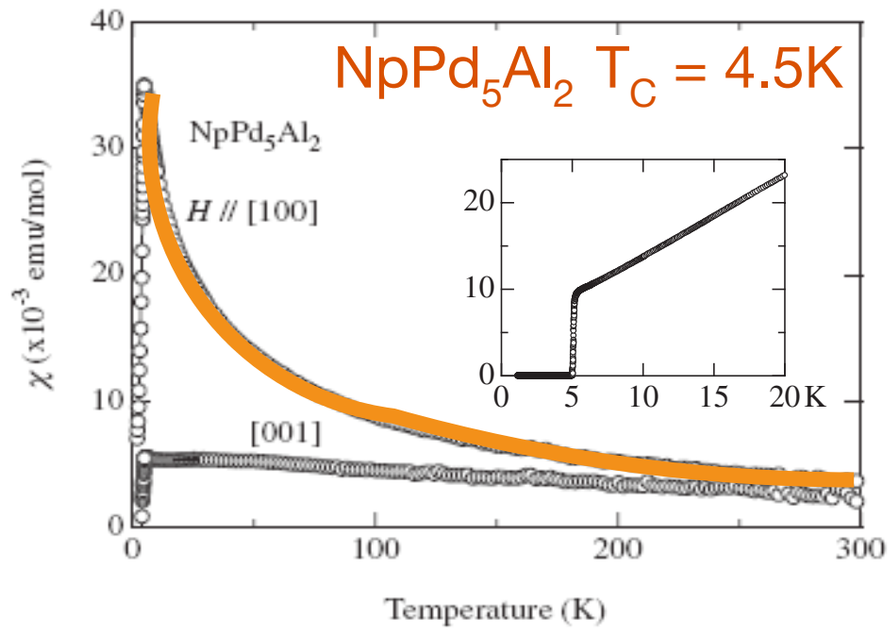

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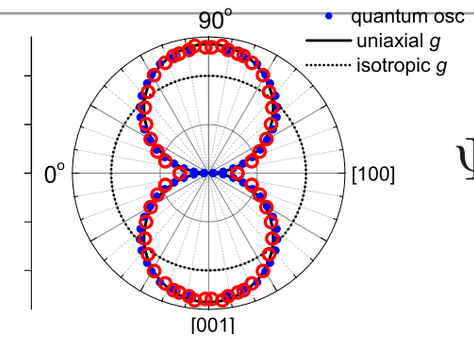

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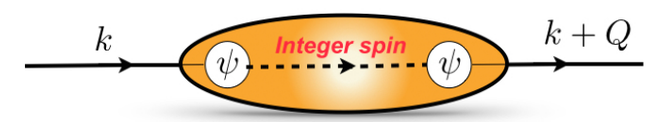

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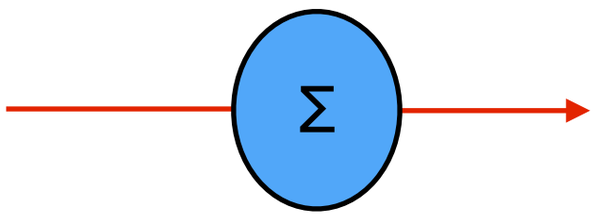
But how can we demonstrate this?

Fractionalization and Dynamic Order

Fractionalization and Dynamic Order

Dyson self-energy



Conventional Broken Symmetry: Local

$$\Sigma_{\alpha\beta}(2,1) = M_{\alpha\beta}\delta(2-1)$$

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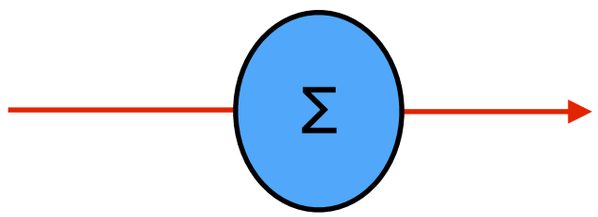

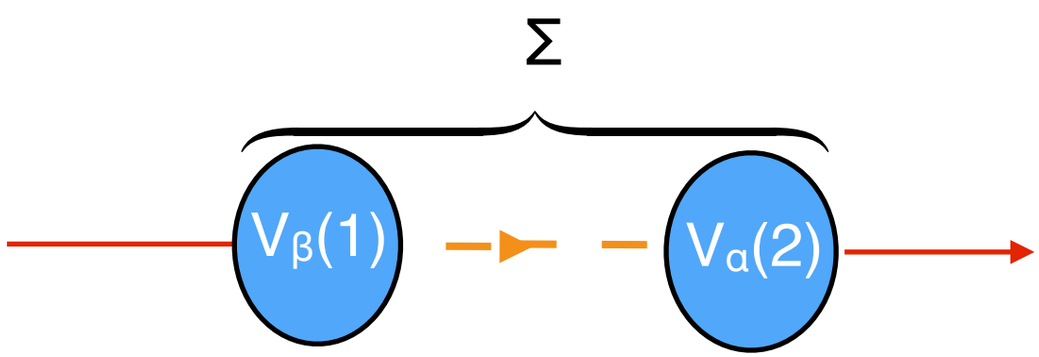

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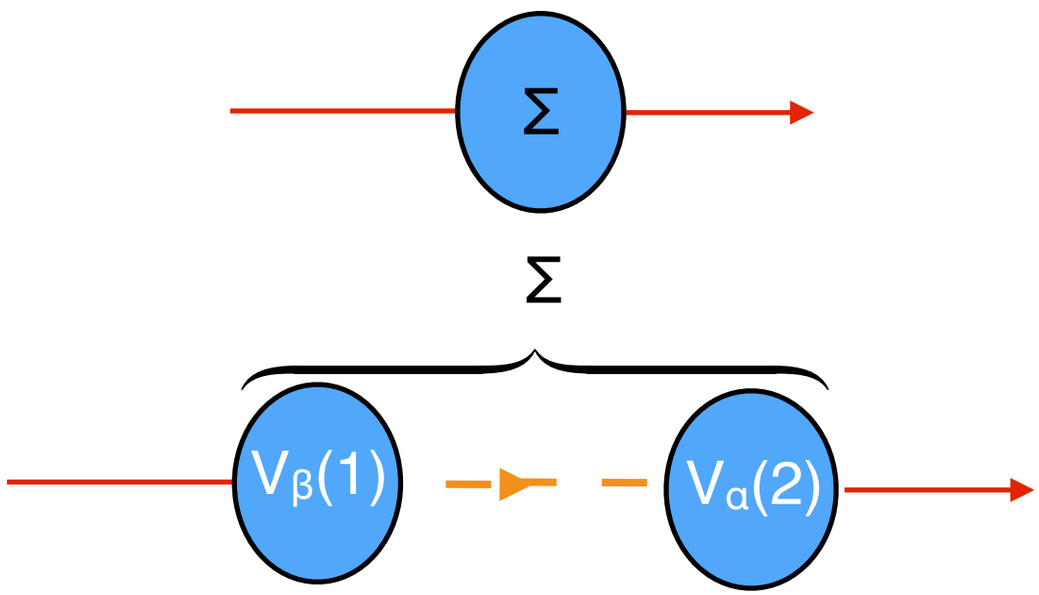

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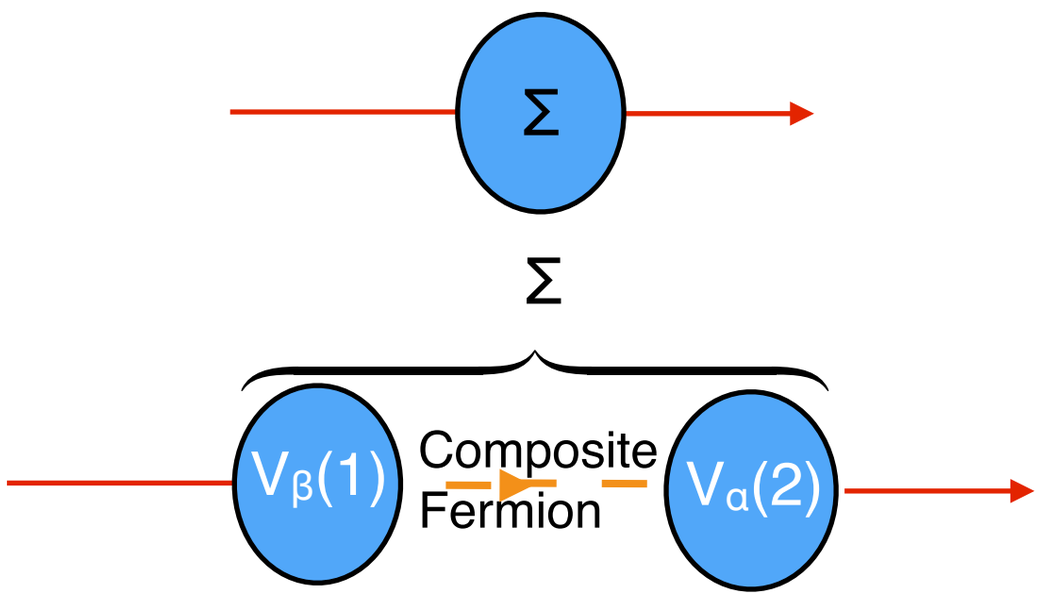

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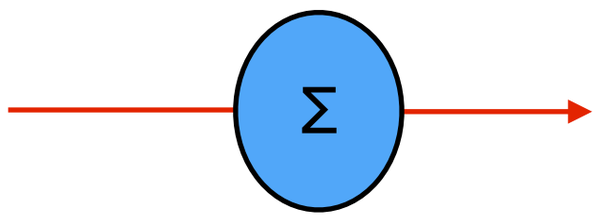

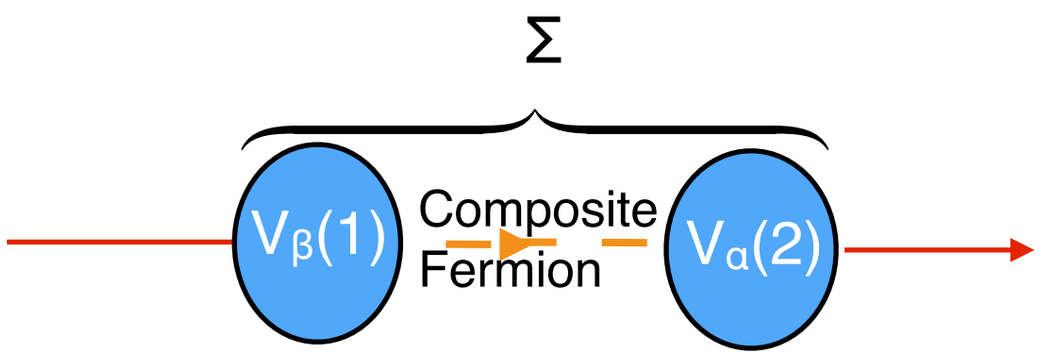

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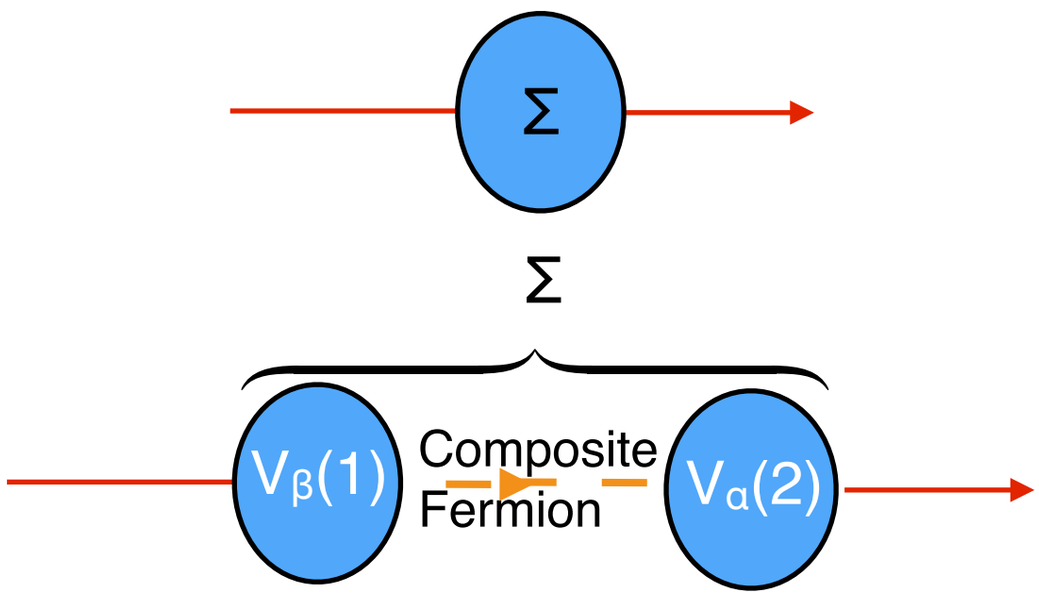

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Sakai, Civelli and Imada
PRL 116, 057003 (2016)
Konik, Rice, Tsvelik (2006)

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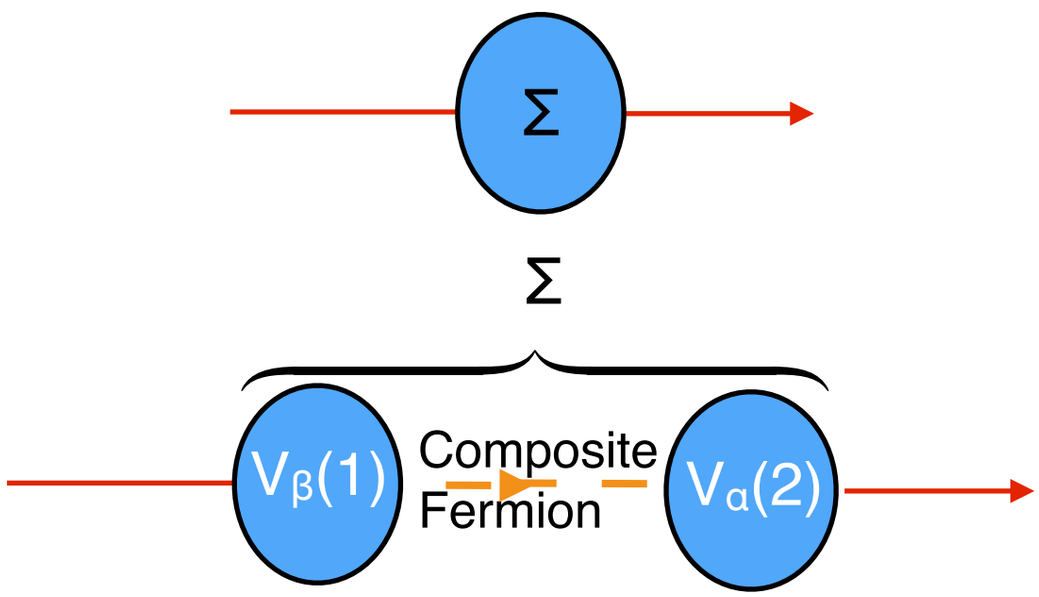

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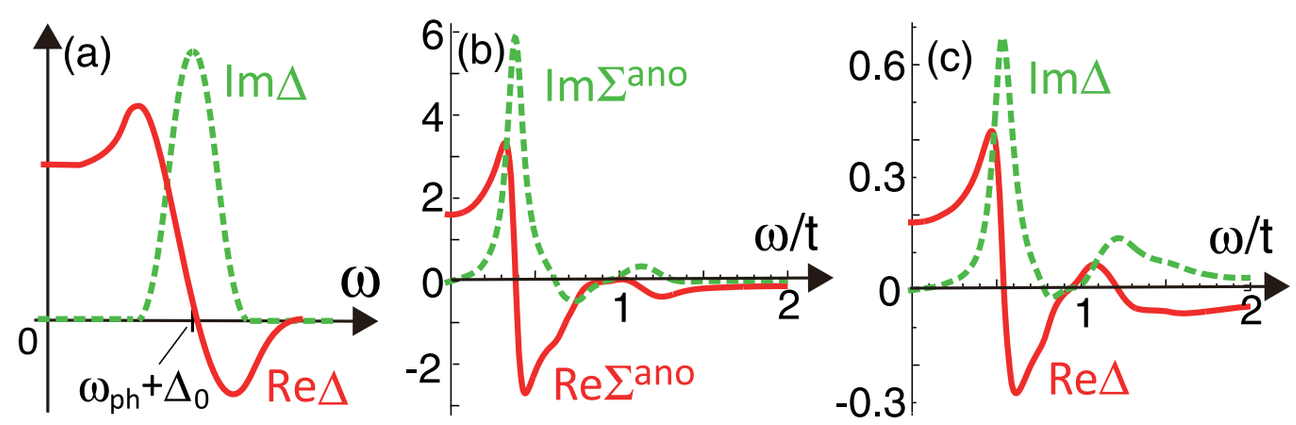
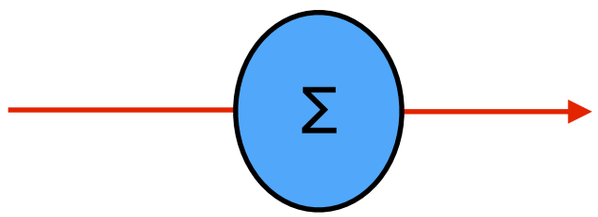
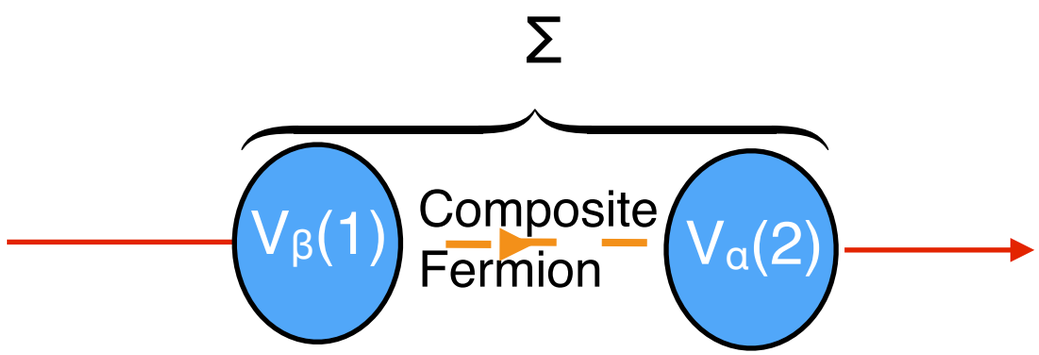


Fractionalization and Dynamic Order

Dyson self-energy





Conventional Broken Symmetry: Local

$$\Sigma_{\alpha\beta}(2,1) = M_{\alpha\beta}\delta(2-1)$$

Order Fractionalization: non-local in time.

$$\Sigma_{\alpha\beta}(2,1) \xrightarrow{|t_2-t_1|\to\infty} V_\alpha(2)V_\beta(1)\mathrm{sgn}(t_2-t_1)$$

Order fractionalization, if it occurs, is linked to the formation of fermionic bound-states

$$(\overline{\psi\psi\psi})_\Lambda(x)$$

Fractionalization and Dynamic Order

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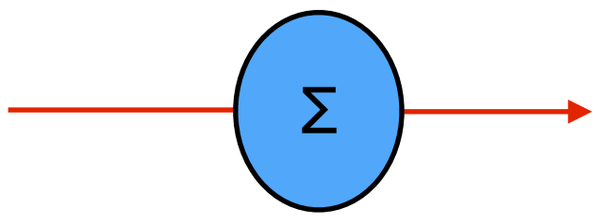

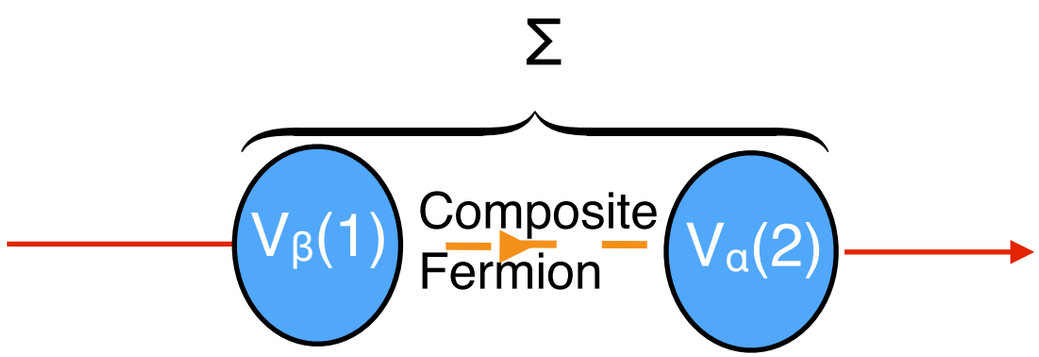

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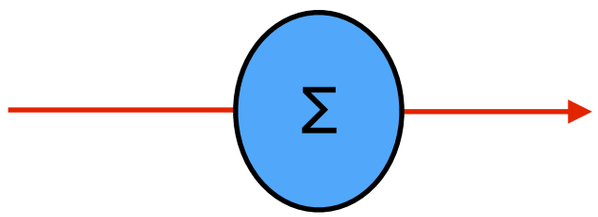

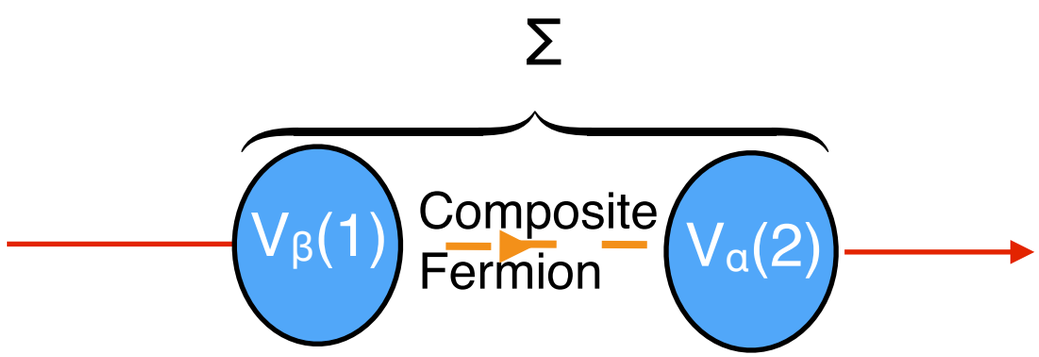

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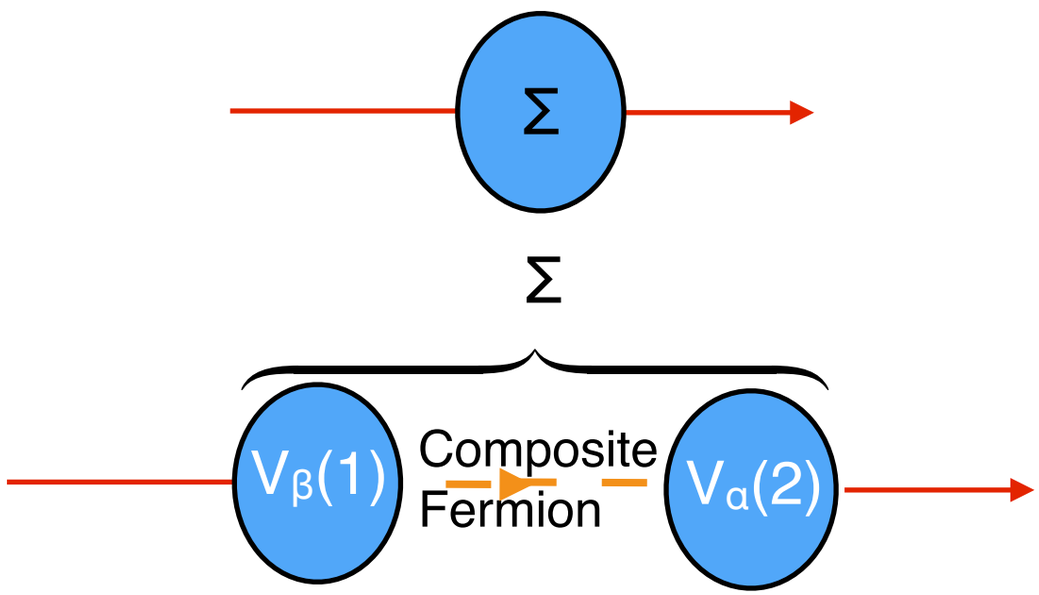

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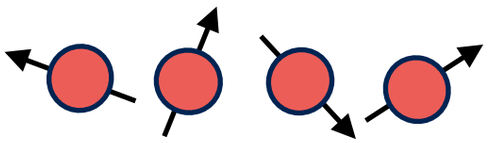
Bound state fractionalizes into order parameter and dark fermion

$$\Lambda = \Big(\{\lambda\},\{\alpha\}\Big)$$

From Weiss to Kondo

Weiss Molecular Field



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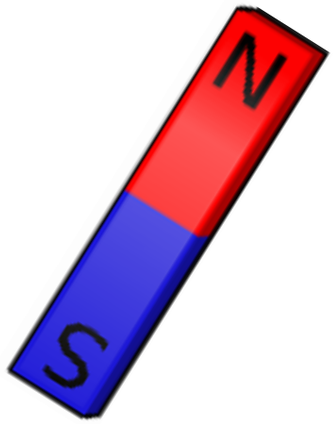

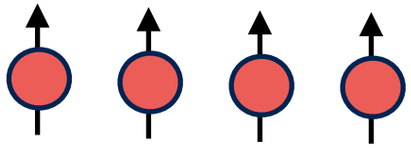

M(B)

Induced Broken Symmetry

From Weiss to Kondo

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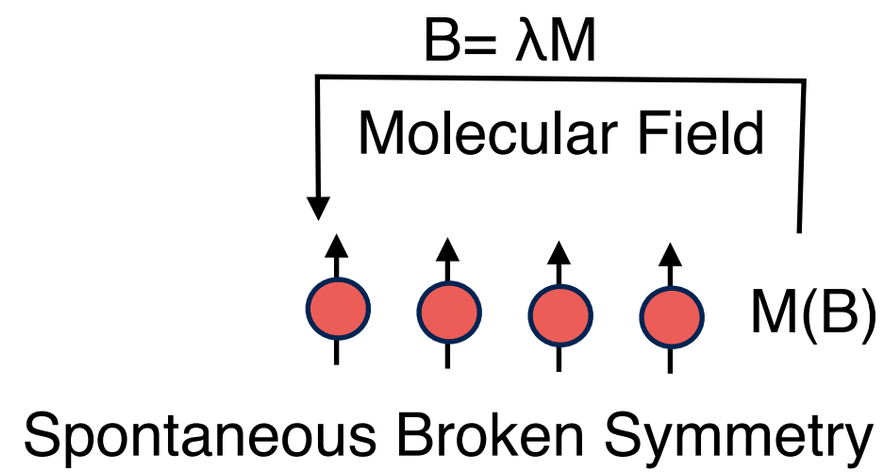
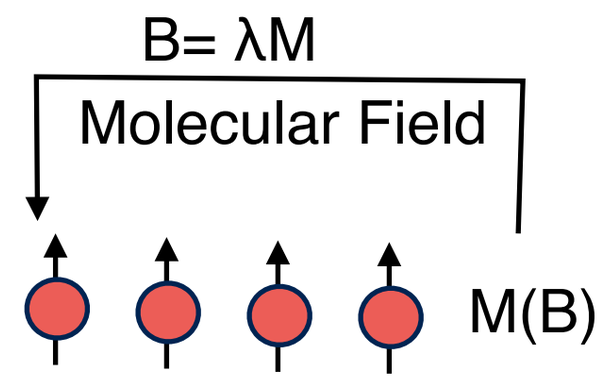
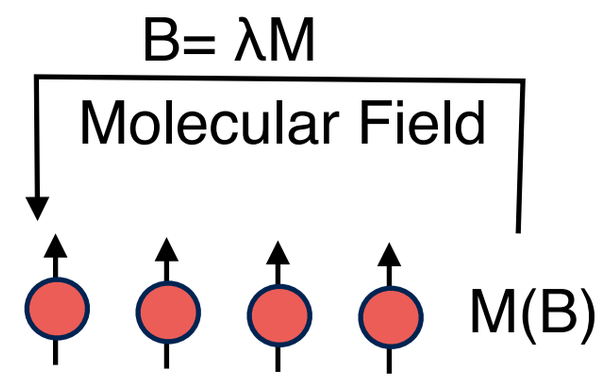


From Weiss to Kondo



Pre-requisite:

From Weiss to Kondo



Pre-requisite:

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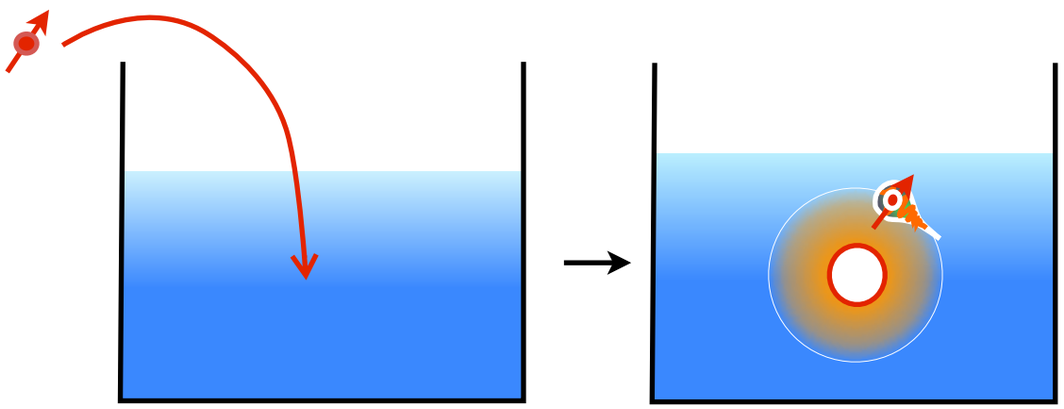
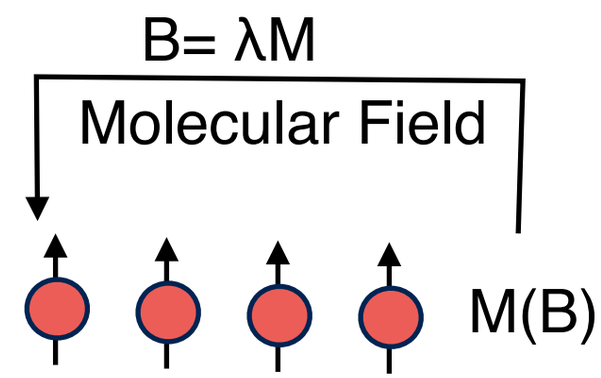
From Weiss to Kondo

Kondo Model: ideal setting



$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + J \psi^{\dagger}_{0} \vec{\sigma} \psi_0 \cdot \vec{S}_0$$



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Fractionalization in the Kondo effect

Fractionalization and Hybridization

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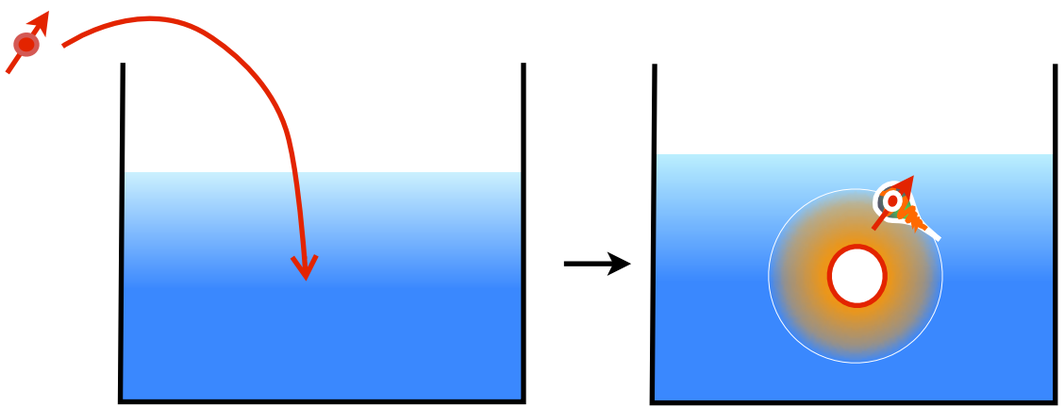

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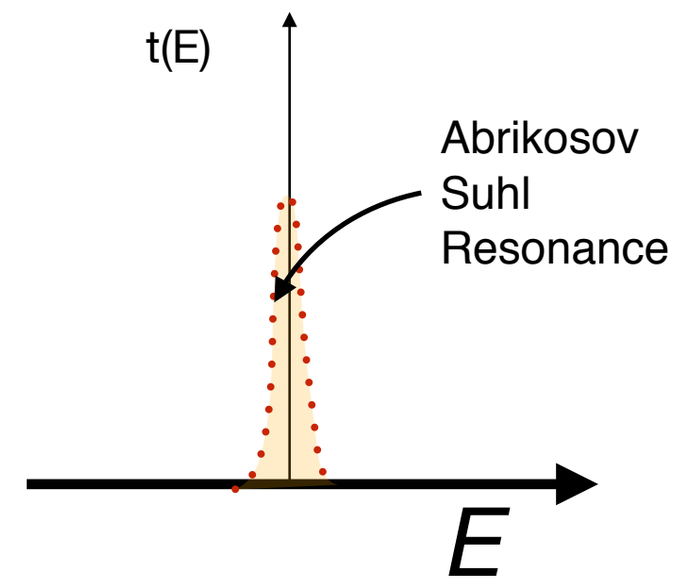

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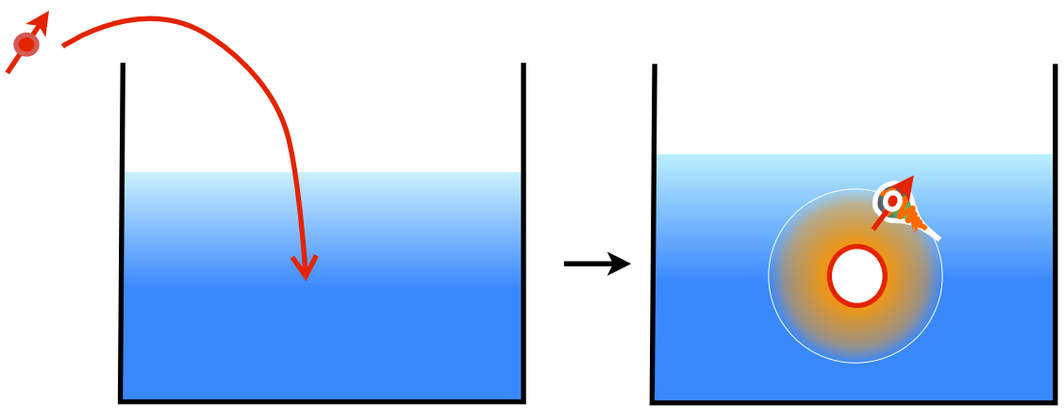

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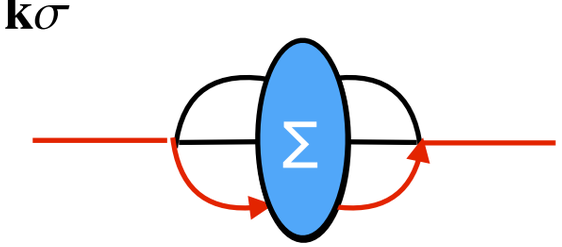
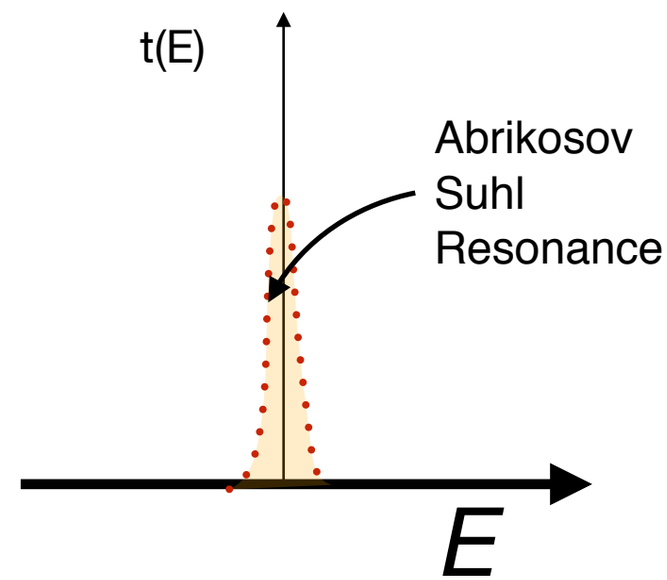


Irreducible self-energy



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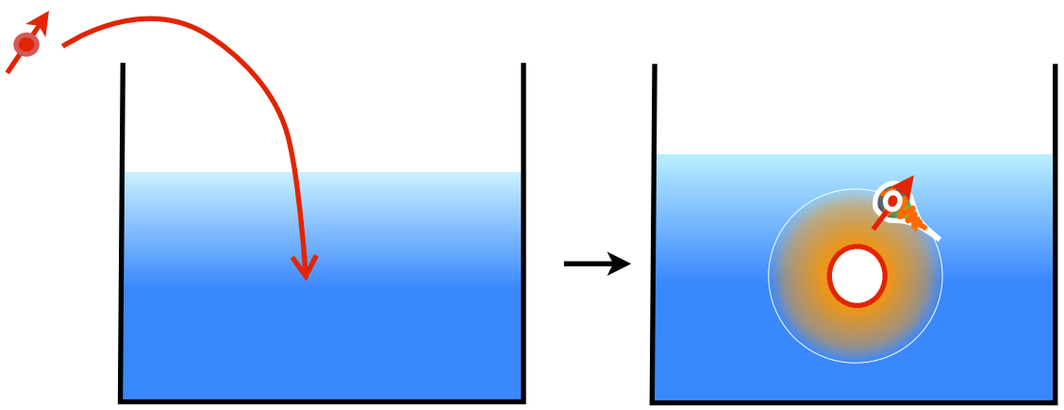

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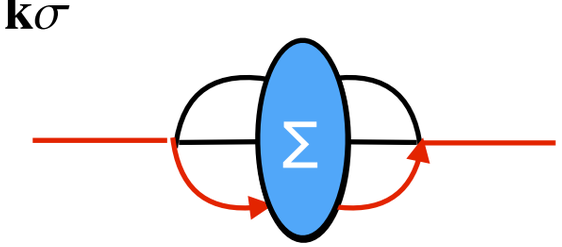

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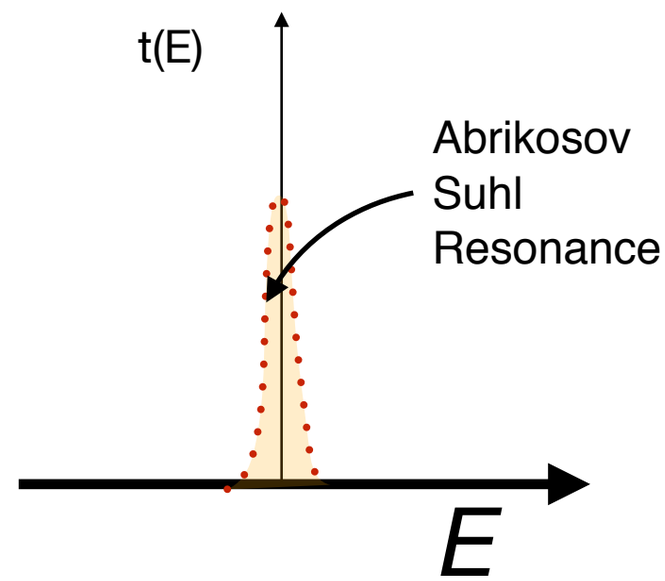
Nozieres Local Fermi Liquid
Large N Mean-Field Theory



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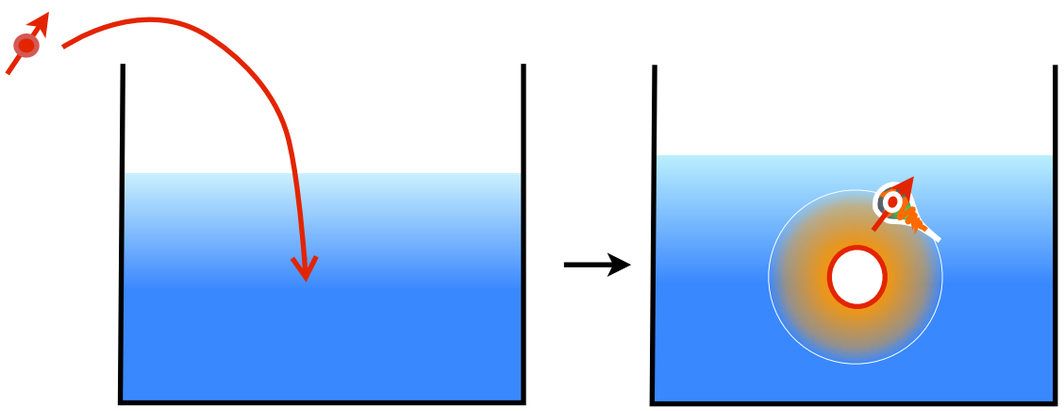

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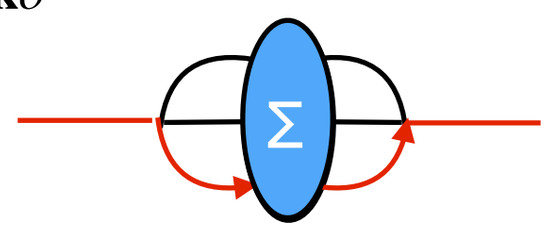

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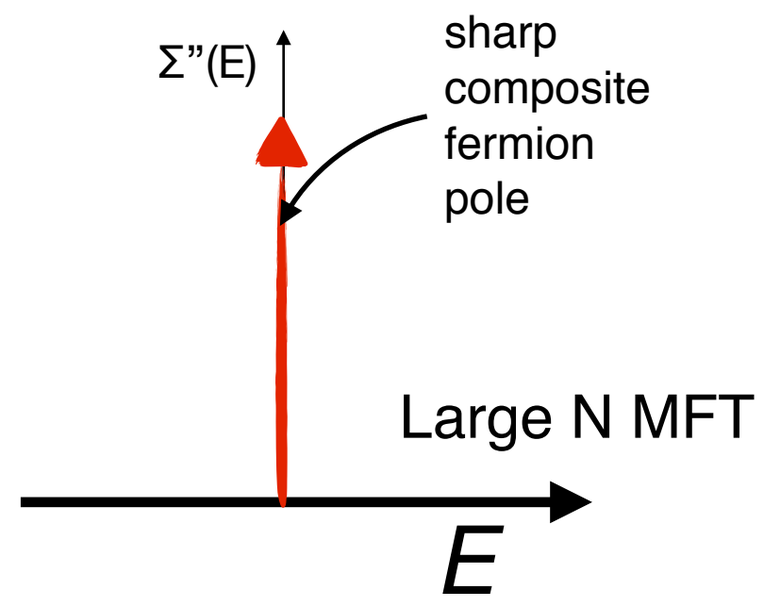

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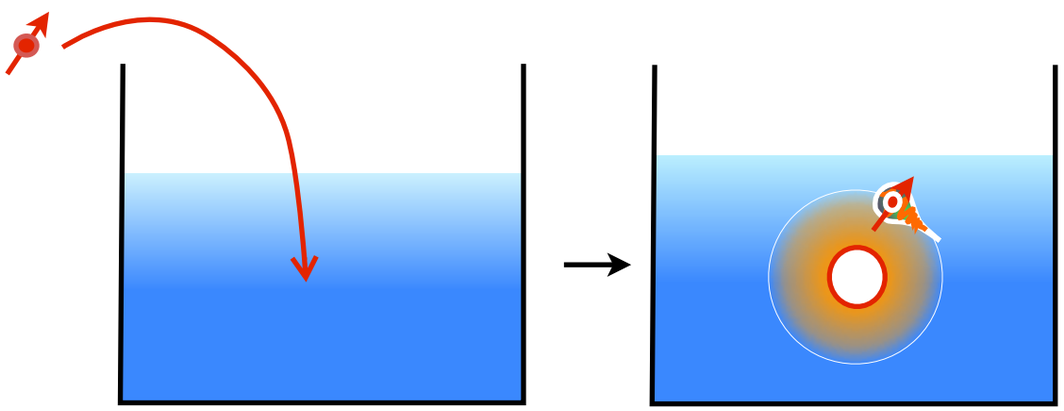

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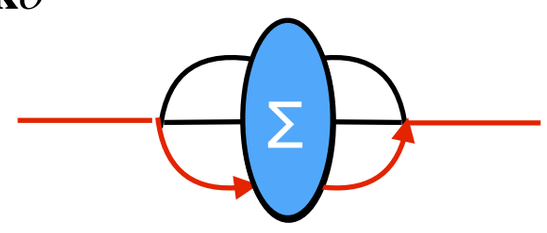

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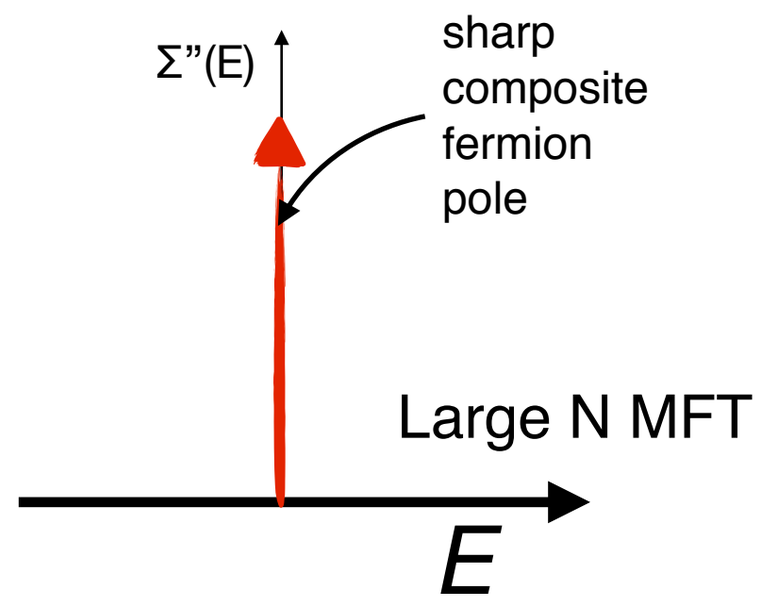

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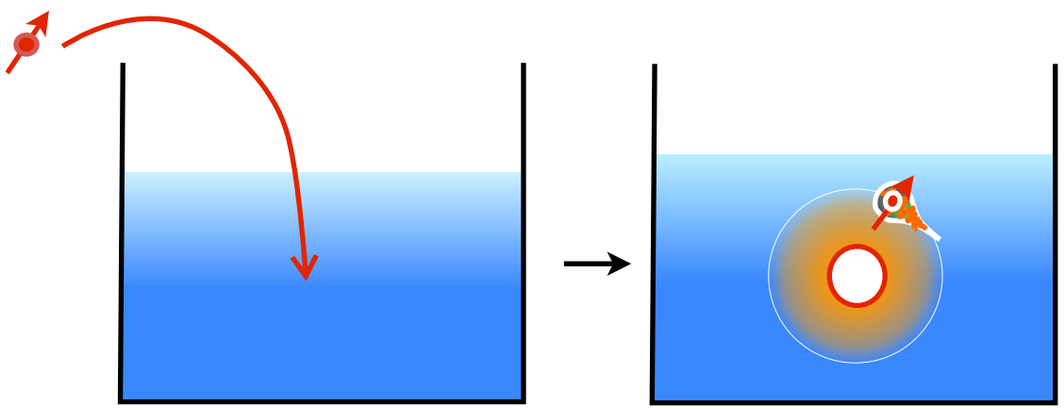

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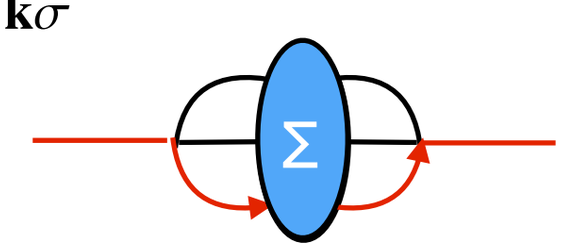

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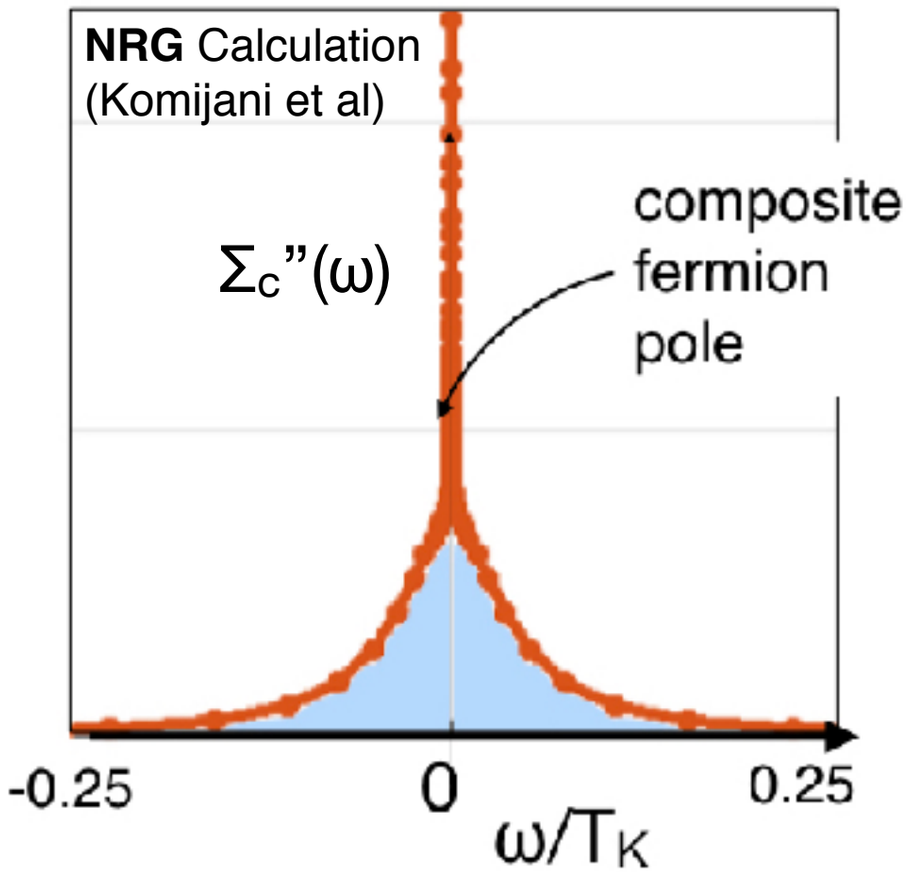

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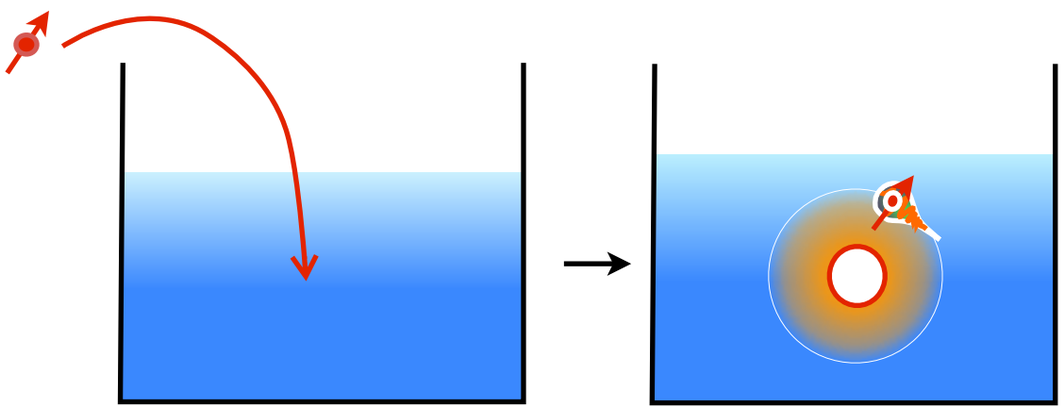

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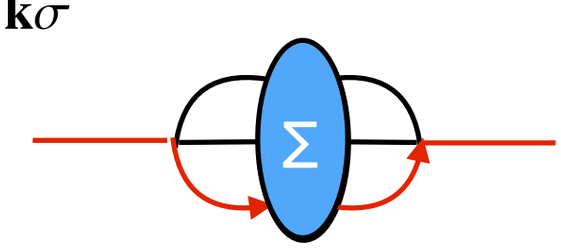

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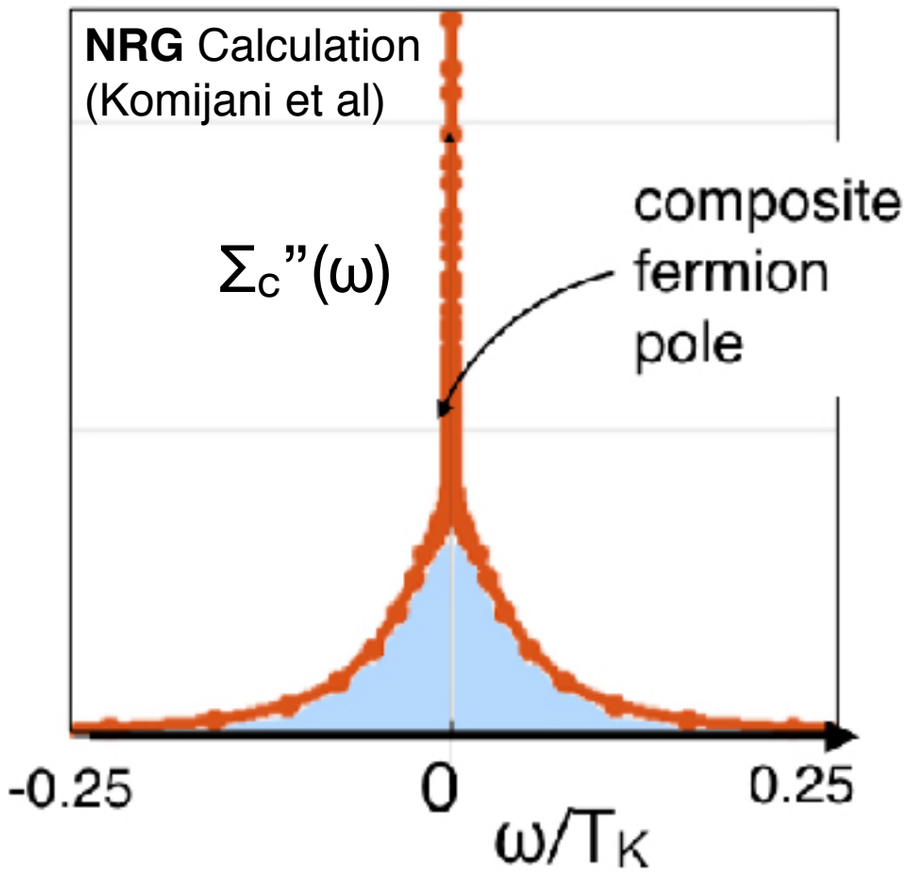

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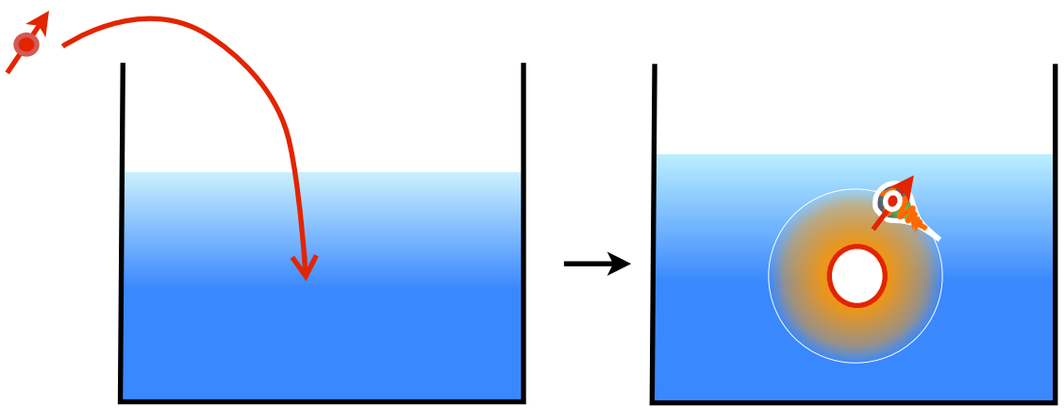

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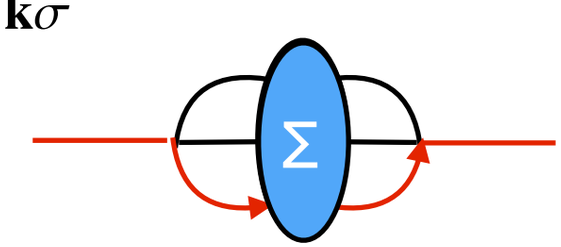

Irreducible t-matrix

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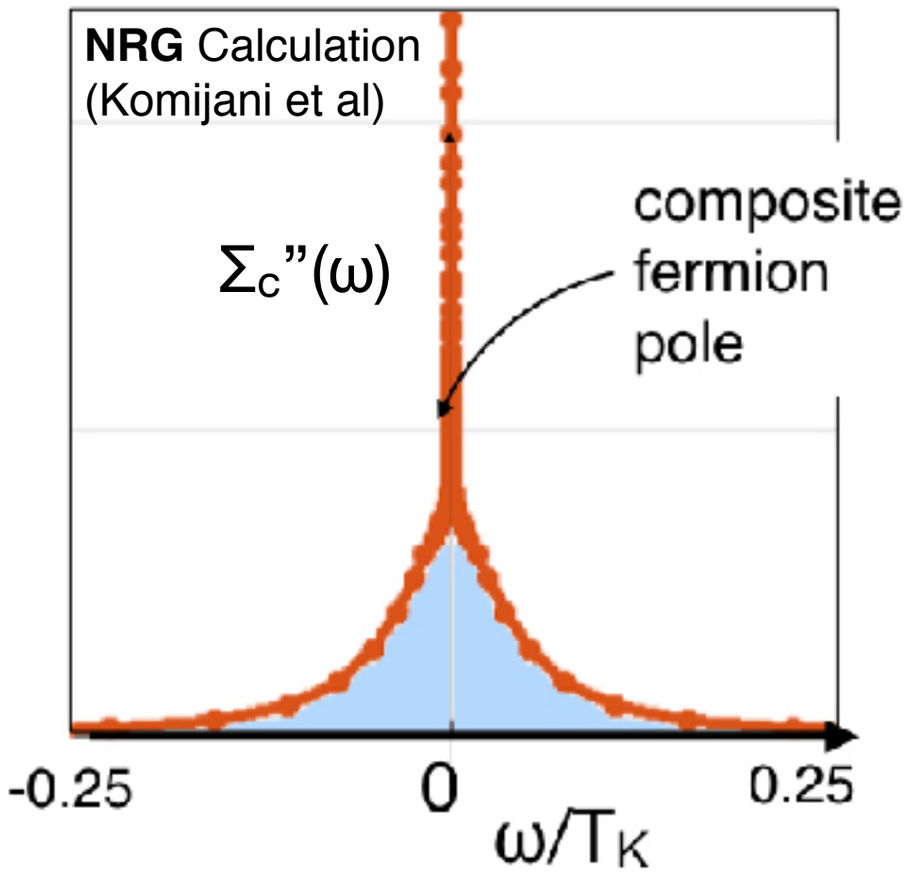

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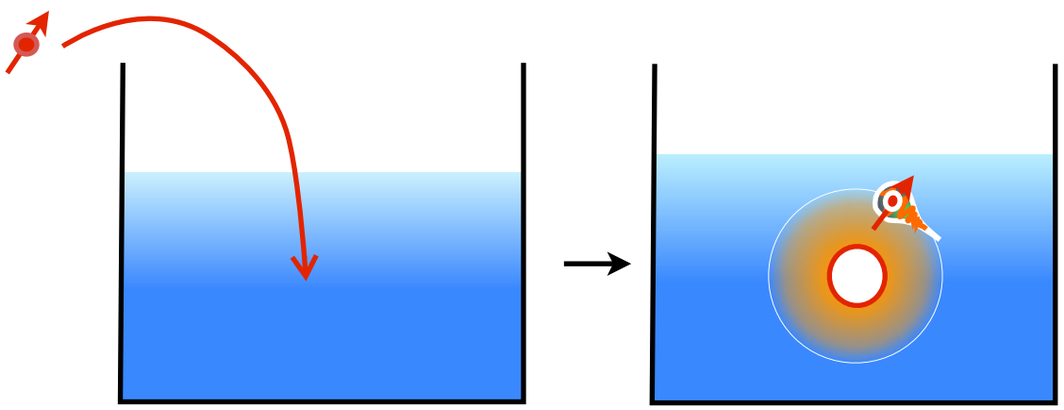

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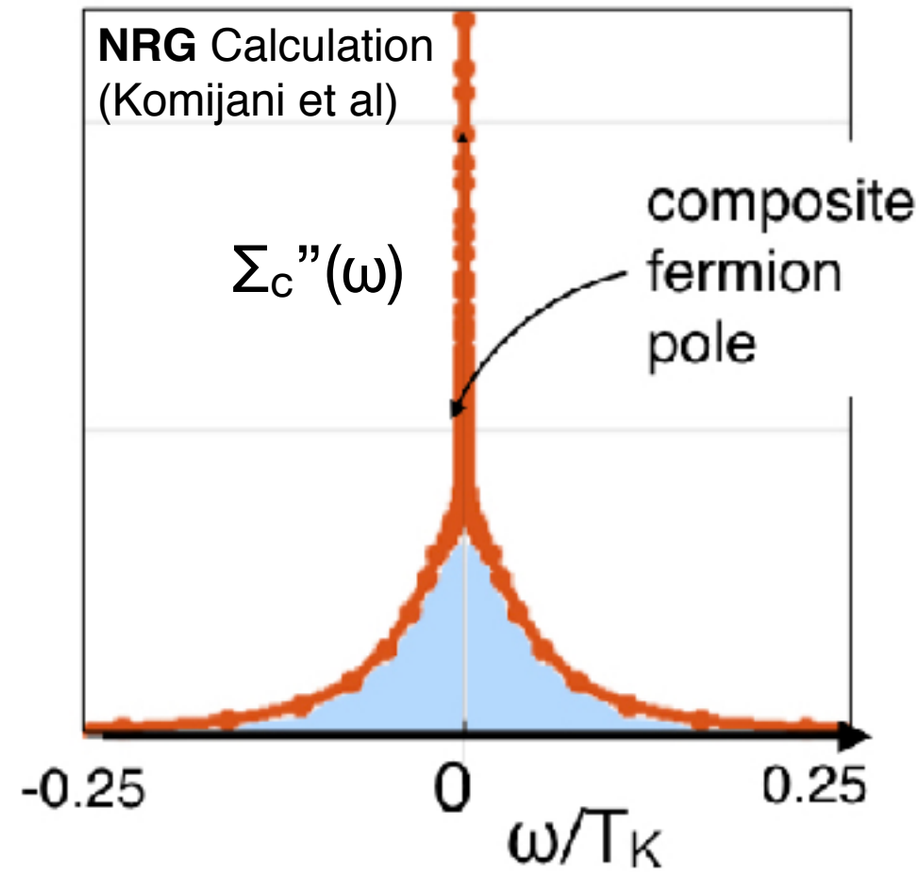

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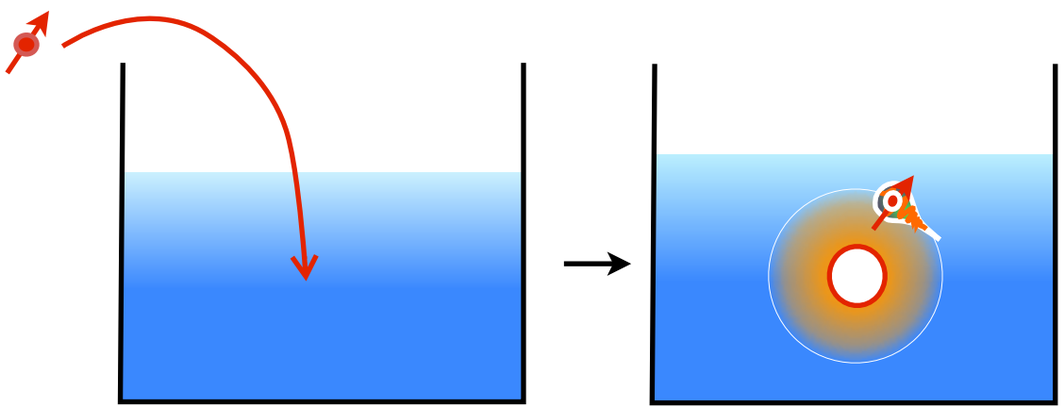

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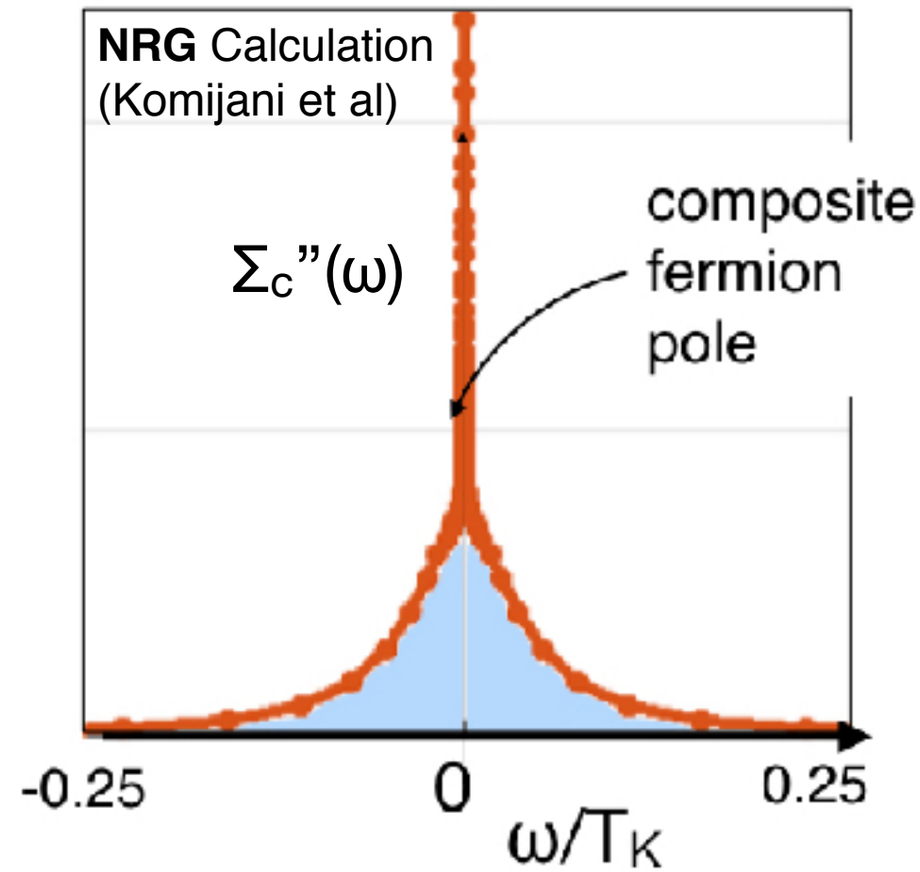

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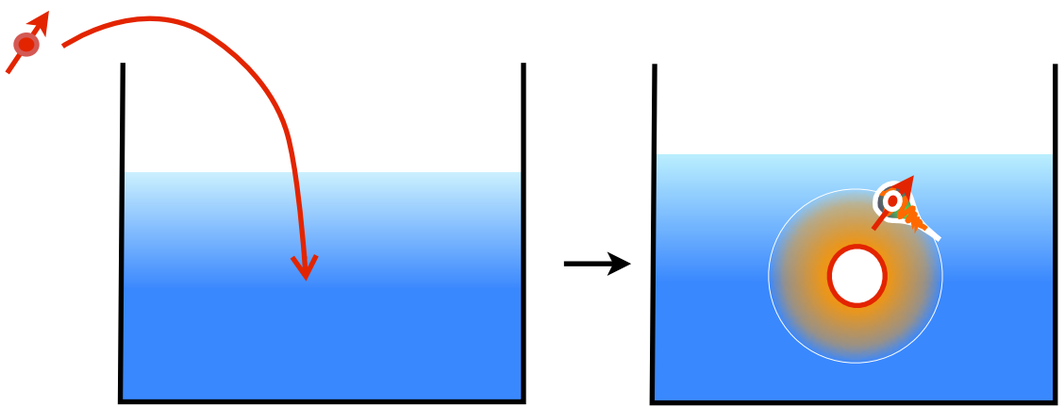

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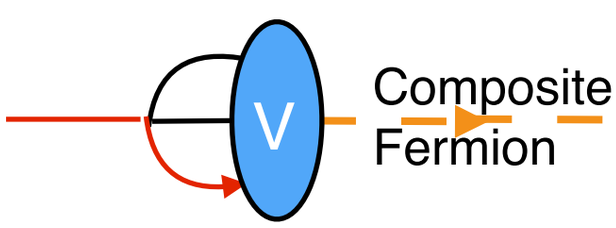
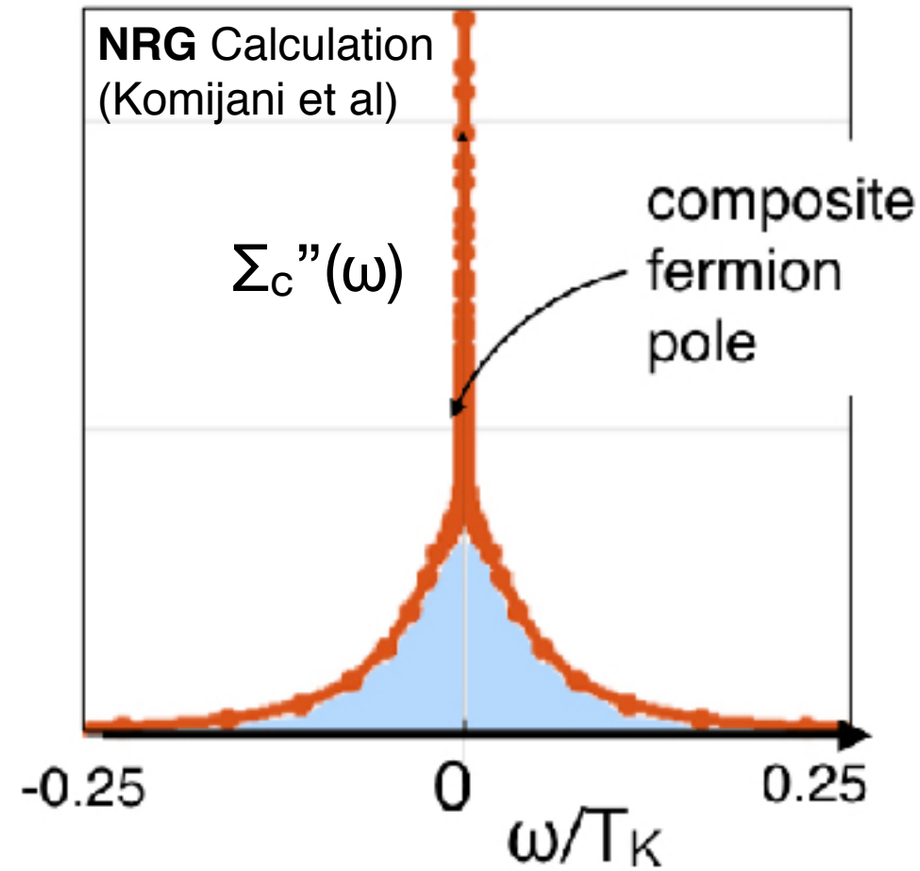
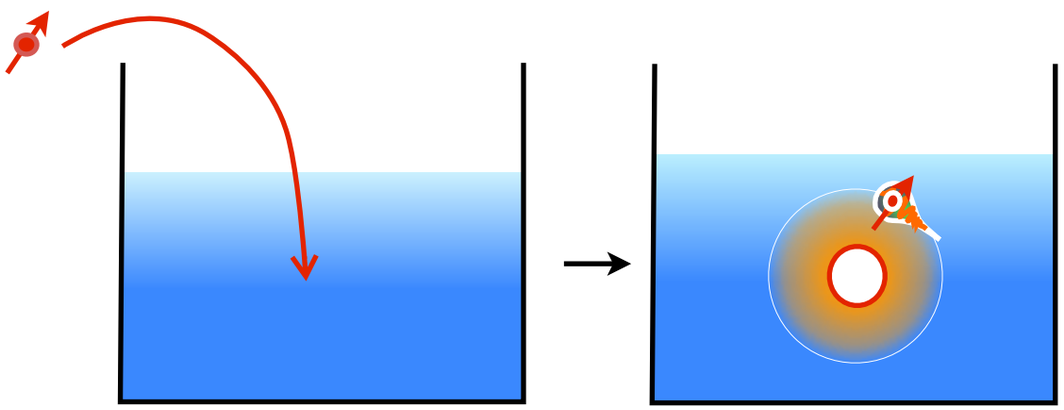




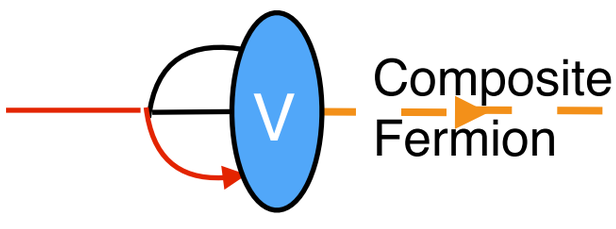
But large N assumes fractionalization,
Does it happen at S=1/2? <u>Confirmed.</u>

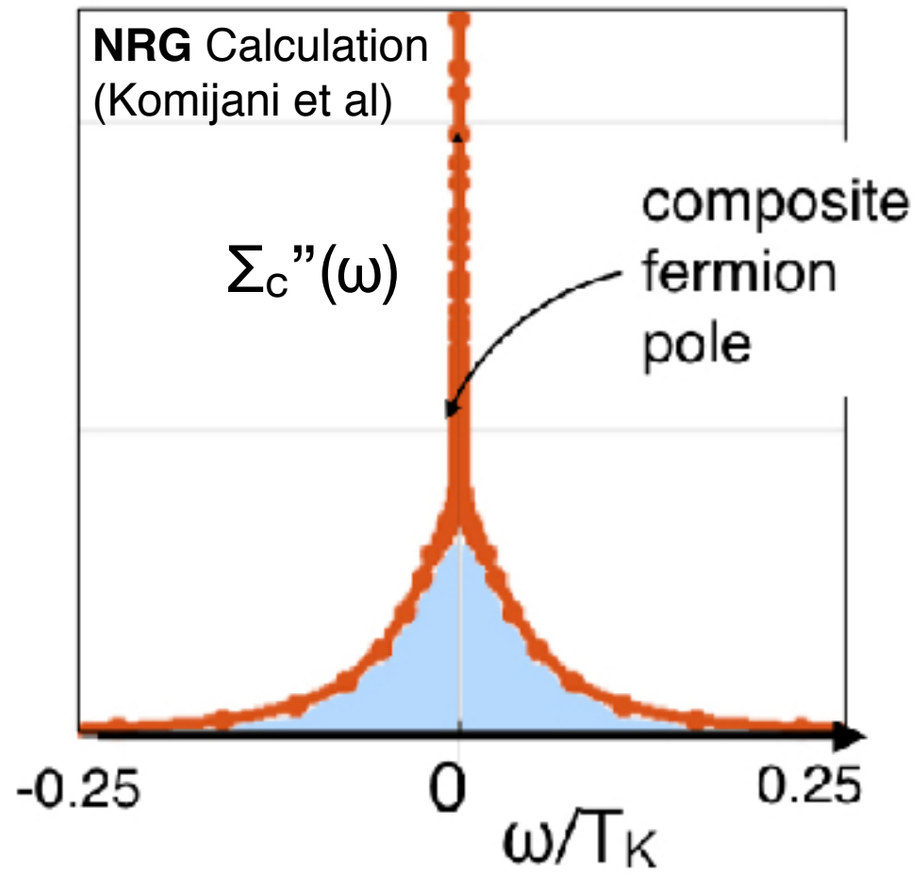
Fractionalization and Hybridization

Kondo Model: ideal setting



$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + J\psi^{\dagger}_0 \vec{\sigma}\psi_0 \cdot \vec{S}_0$$





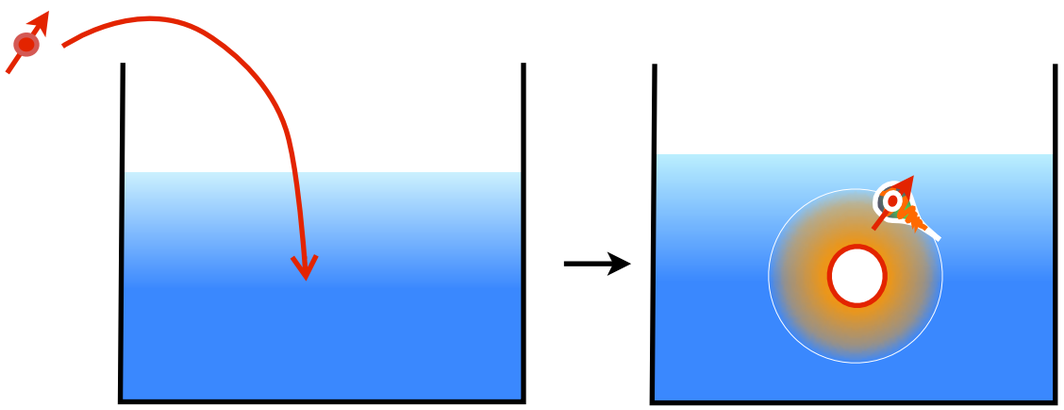
But large N assumes fractionalization,
Does it happen at S=1/2? <u>Confirmed.</u>

$$\mathcal{F}_\alpha = J(\vec{\sigma}\cdot\vec{S}_0)\psi_{0\alpha} \rightarrow V f_\alpha(0)$$

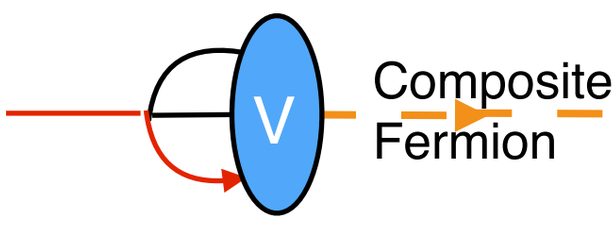
<u>*Three-body*</u> Bound State

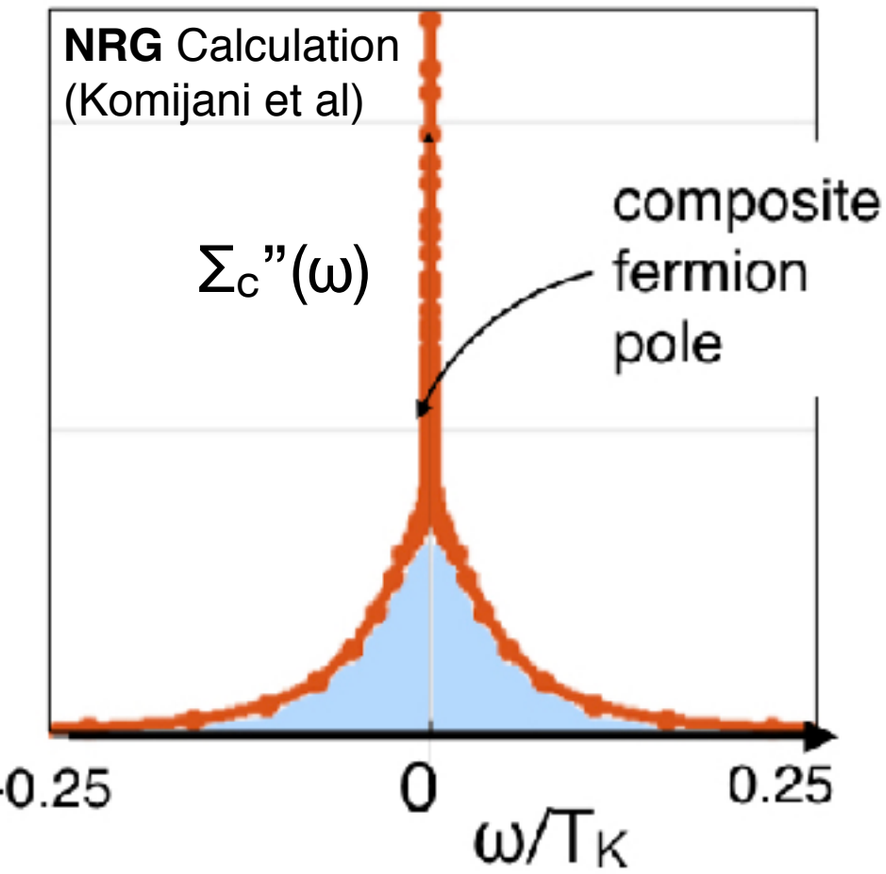
Fractionalization and Hybridization

Kondo Model: ideal setting



$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + J \psi^{\dagger}_0 \vec{\sigma} \psi_0 \cdot \vec{S}_0$$





But large N assumes fractionalization,
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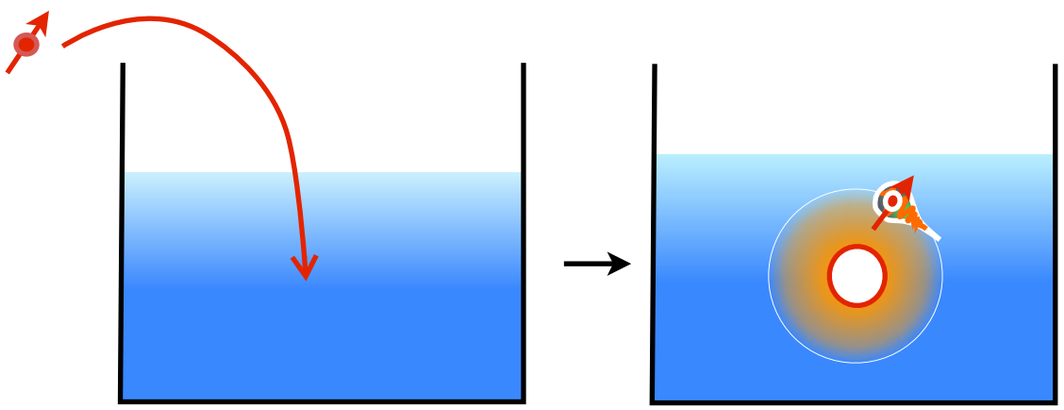
$$\mathcal{F}_\alpha = J(\vec{\sigma} \cdot \vec{S}_0)\psi_{0\alpha} \rightarrow V f_\alpha(0)$$

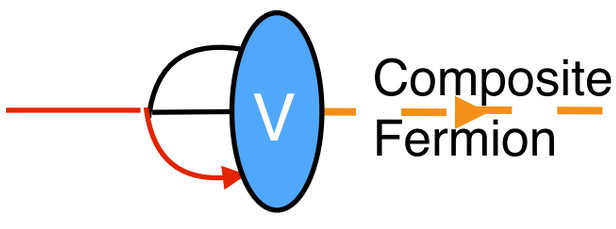
Three-body Bound State

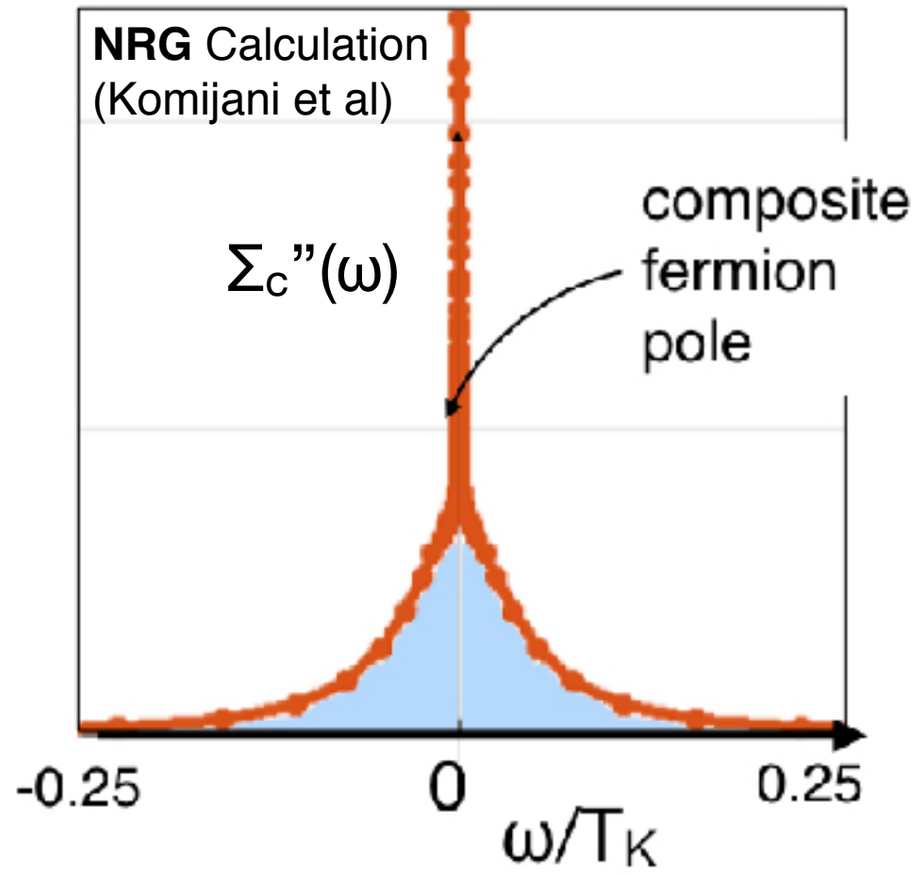
Fractionalization and Hybridization

Kondo Model: ideal setting



$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + V(\psi^{\dagger}_{\sigma} f_{\sigma} + \text{H.c})$$





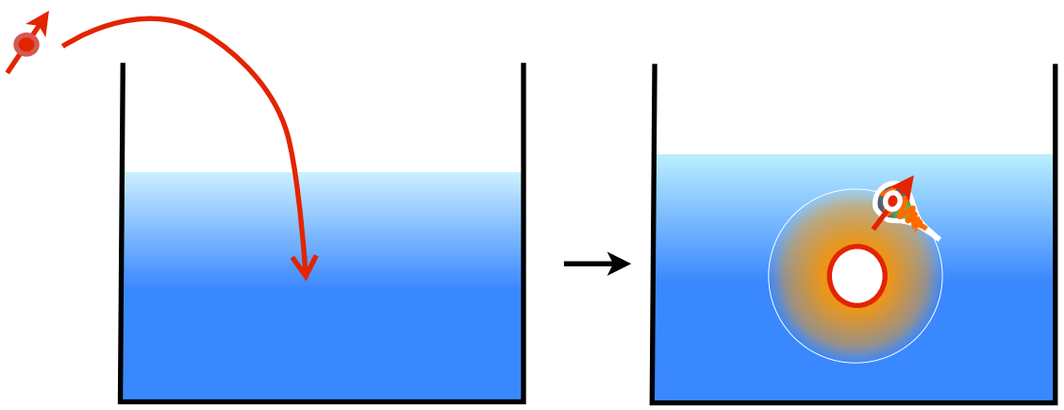
But large N assumes fractionalization,
Does it happen at S=1/2? Confirmed.

$$\mathcal{F}_{\alpha} = J(\vec{\sigma} \cdot \vec{S}_0)\psi_{0\alpha} \rightarrow V f_{\alpha}(0)$$

Three-body Bound State

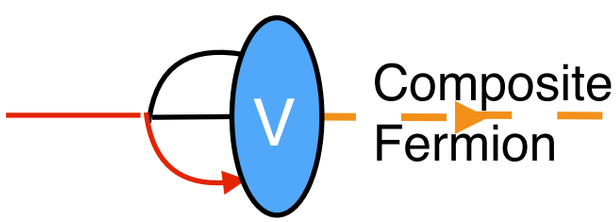
Fractionalization and Hybridization

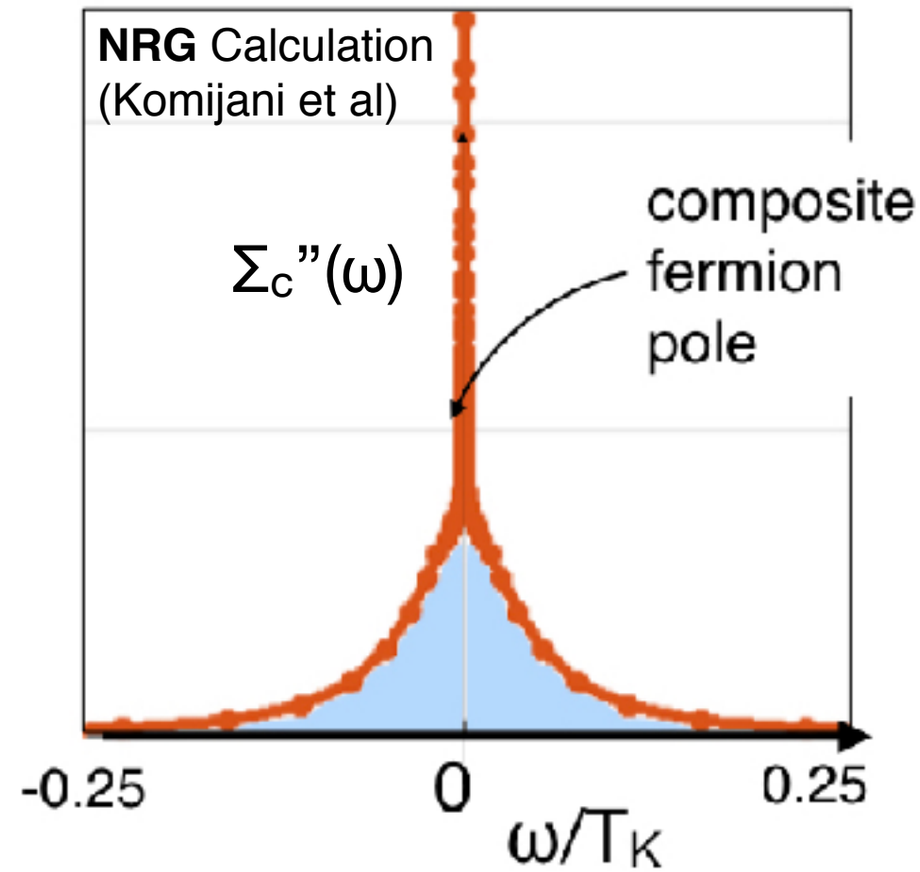
Kondo Model: ideal setting



$$H = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + V(\psi^{\dagger}_{\sigma} f_{\sigma} + \text{H.c})$$

The hybridization is a Higgs field for the spinon which pins its internal U(1) gauge field to the external EM field, giving the f-electrons physical charge.





But large N assumes fractionalization,
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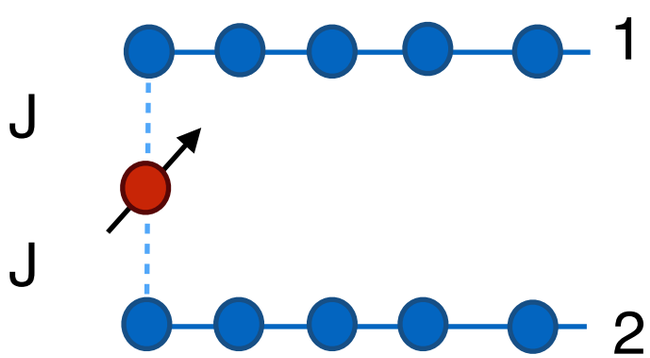
$$\mathcal{F}_\alpha = J(\vec{\sigma}\cdot\vec{S}_0)\psi_{0\alpha} \rightarrow V f_\alpha(0)$$

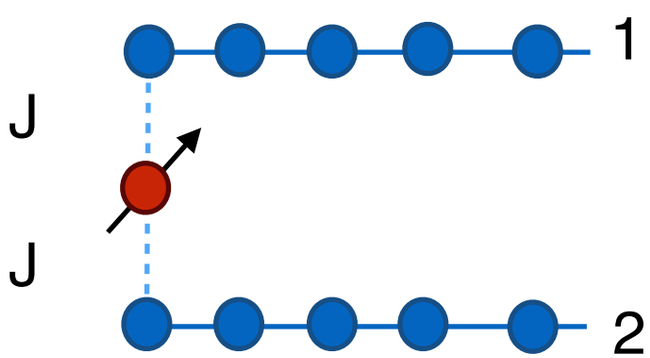
Three-body Bound State

Order Parameter Fractionalization (Induced)



2-channel Kondo

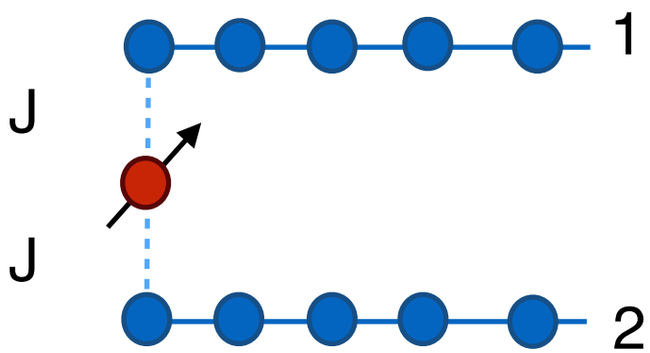
Order Parameter Fractionalization (Induced)



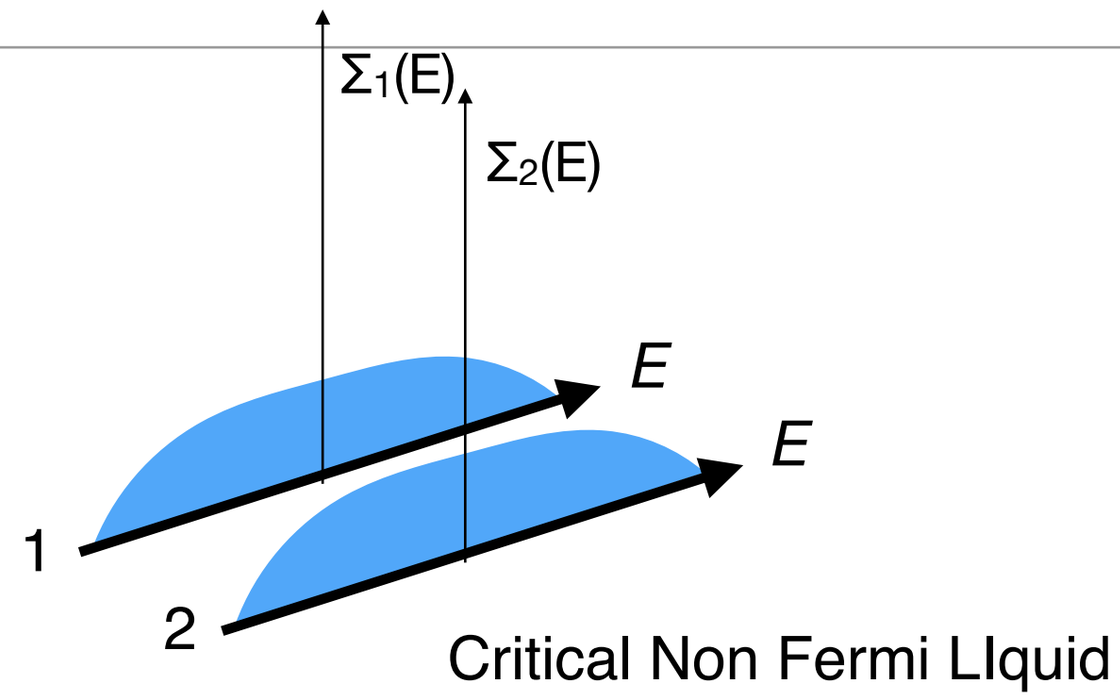
2-channel Kondo

$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\ \sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S}$$

Order Parameter Fractionalization (Induced)



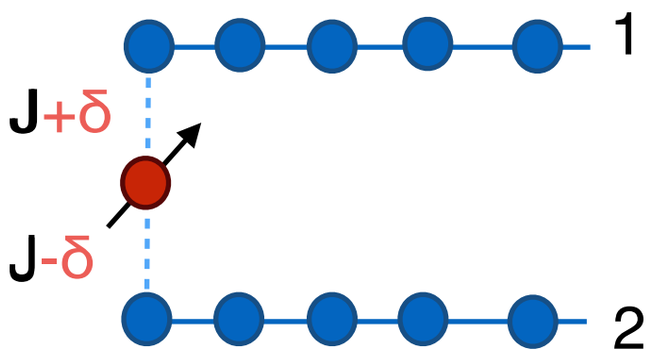
2-channel Kondo



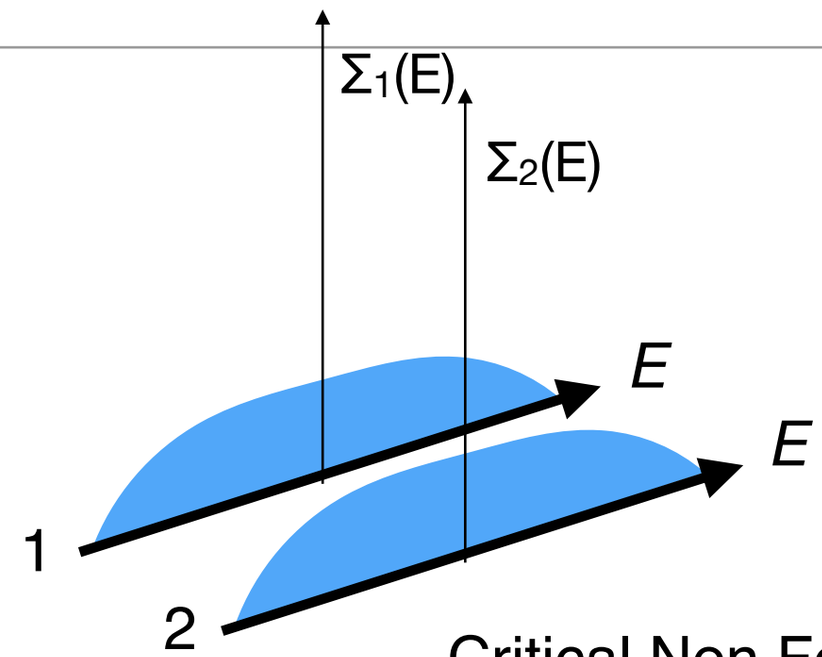
Critical Non Fermi LIquid

$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\bar{\vec{k}}\ \sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S}$$

Order Parameter Fractionalization (Induced)



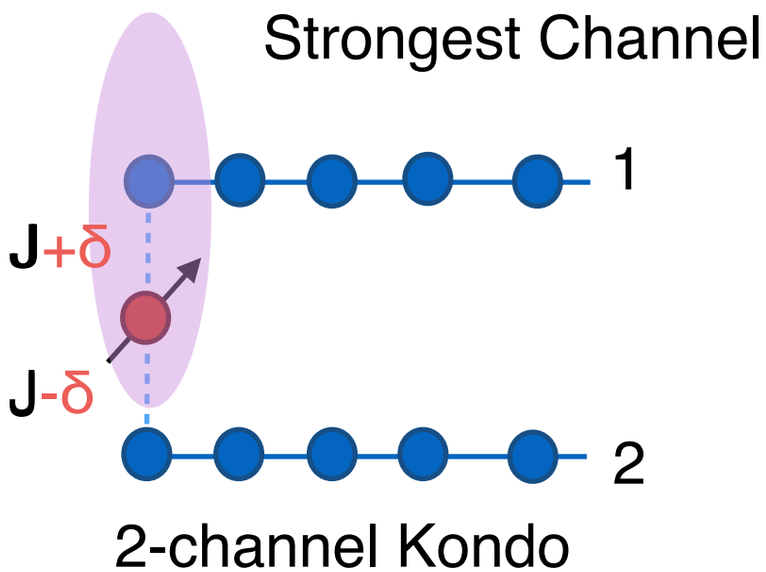
2-channel Kondo

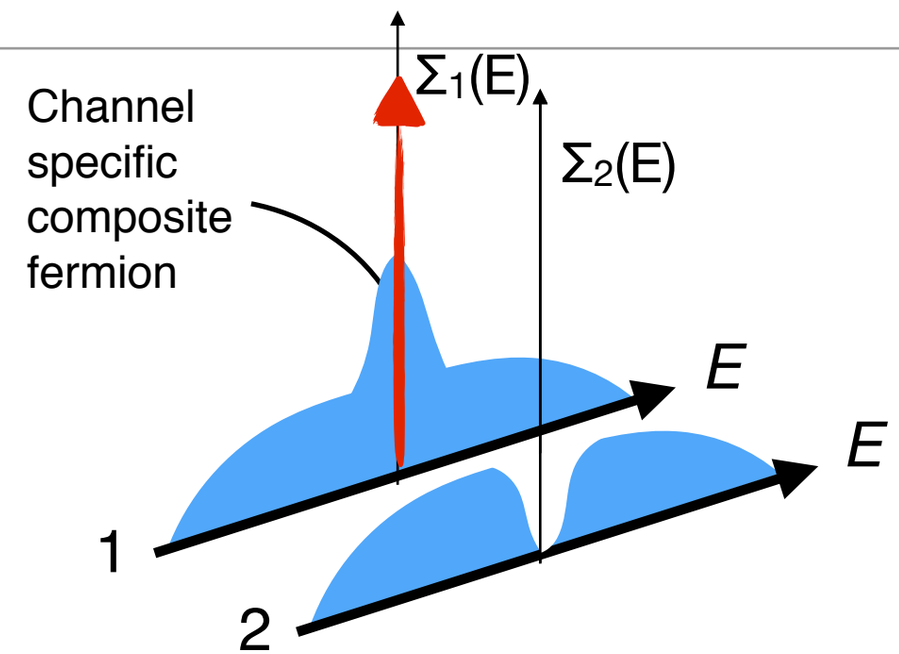


Critical Non Fermi LIquid

$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\ \sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S} + \delta J [\vec{\sigma}_1(0) - \vec{\sigma}_2(0)] \cdot \vec{S}$$

Order Parameter Fractionalization (Induced)





$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\ \sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S} + \delta J [\vec{\sigma}_1(0) - \vec{\sigma}_2(0)] \cdot \vec{S}$$

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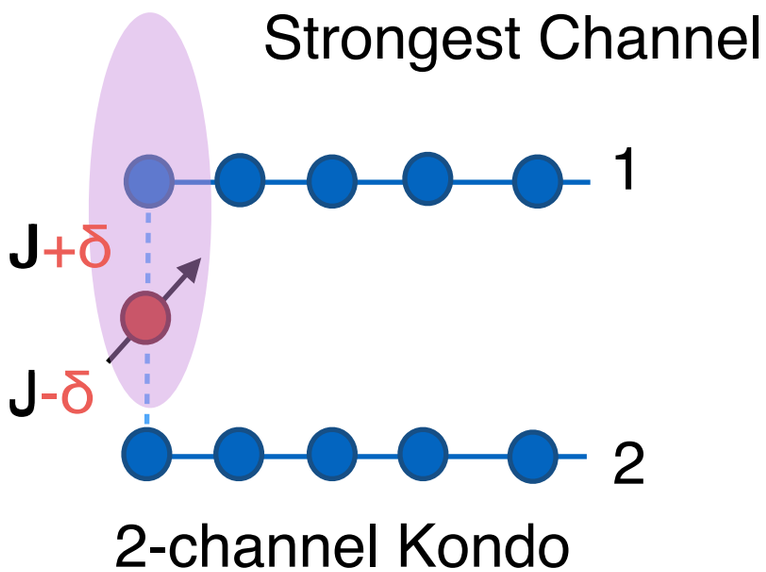
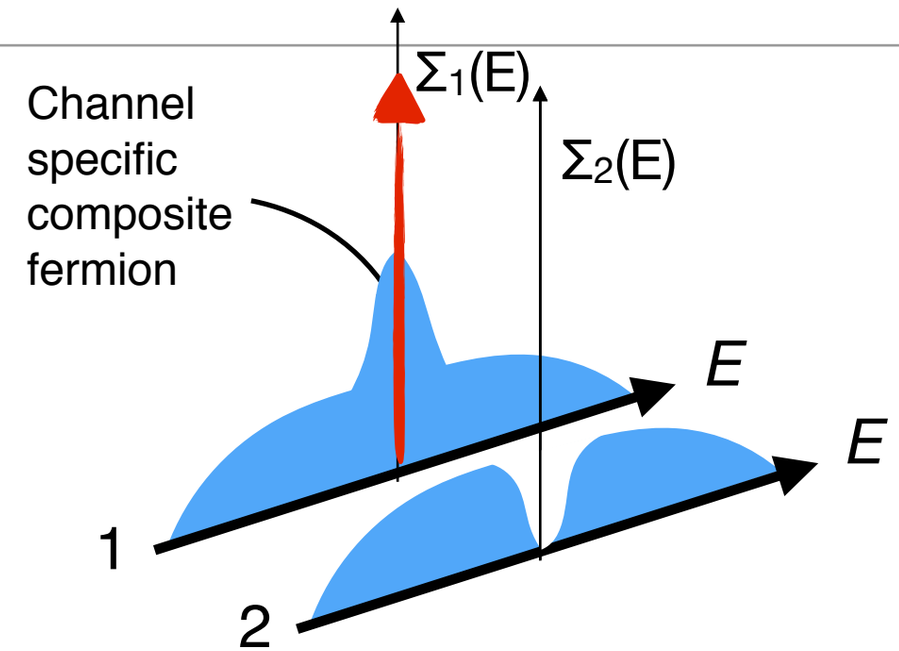
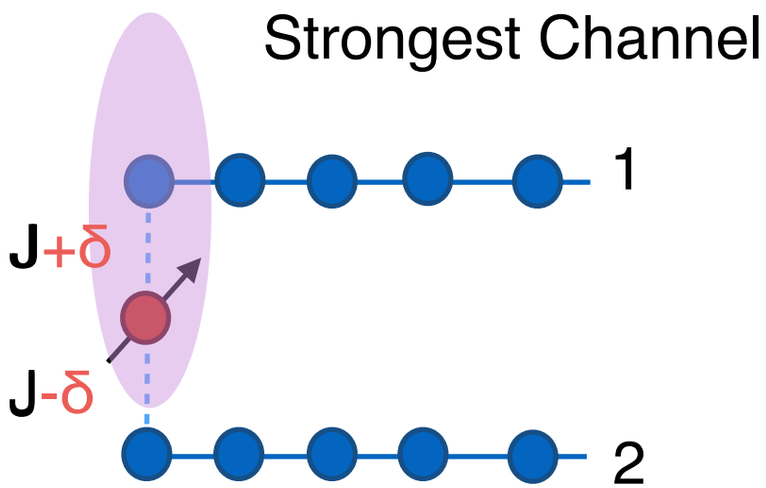
$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\lambda\sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S} + \delta J [\vec{\sigma}_1(0) - \vec{\sigma}_2(0)] \cdot \vec{S}$$



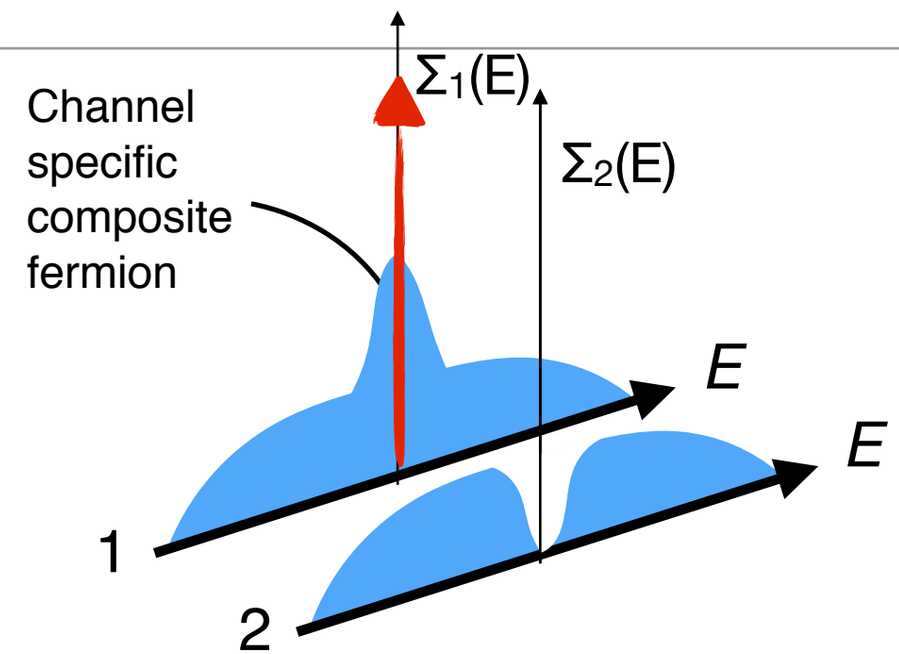


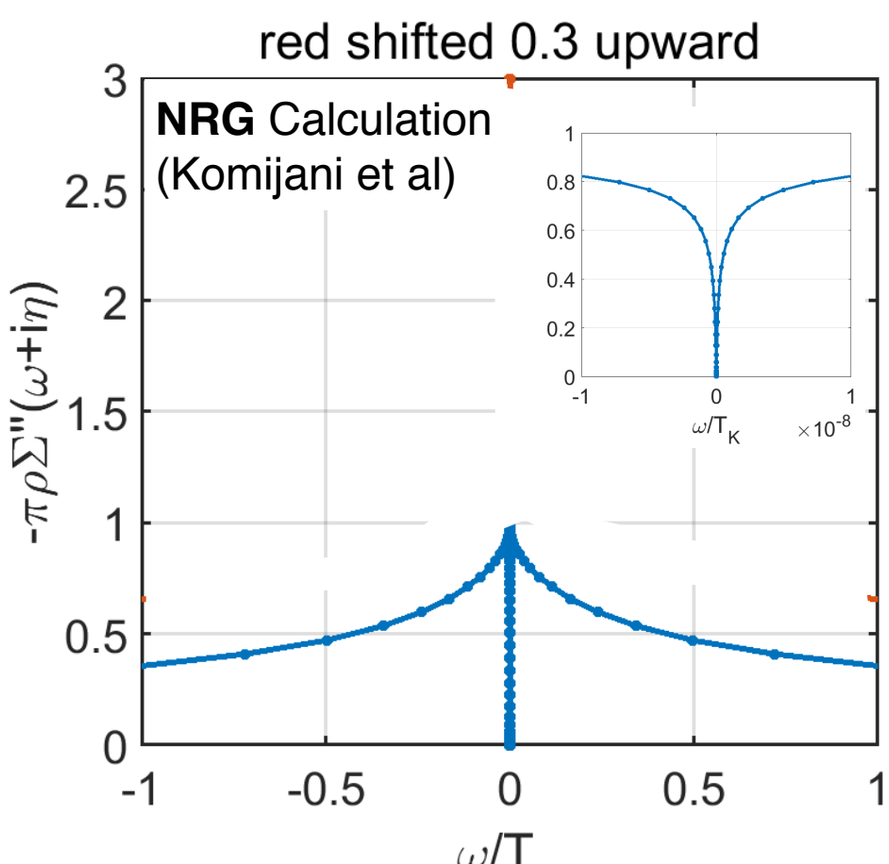
Order Parameter Fractionalization (Induced)

$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\lambda\sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0) \cdot \vec{S} + \delta J [\vec{\sigma}_1(0) - \vec{\sigma}_2(0)] \cdot \vec{S}$$



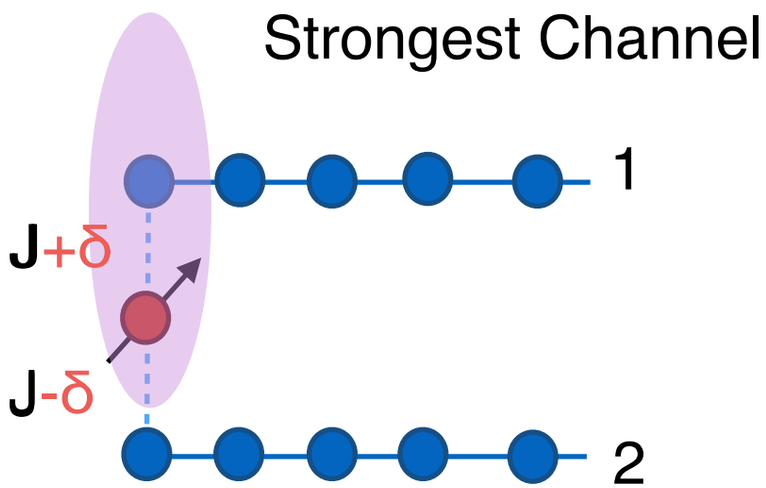
2-channel Kondo



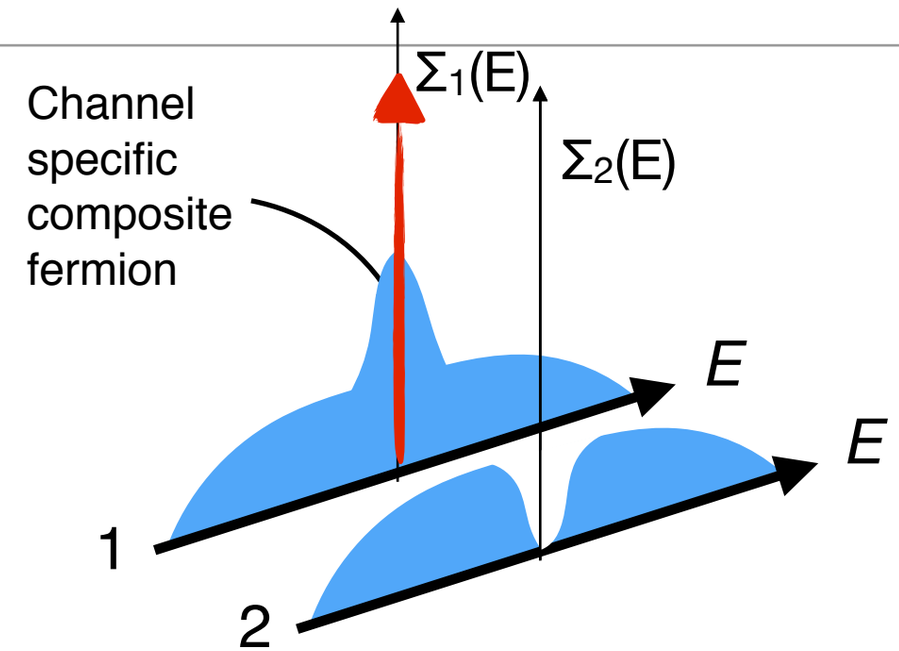


Order Parameter Fractionalization (Induced)

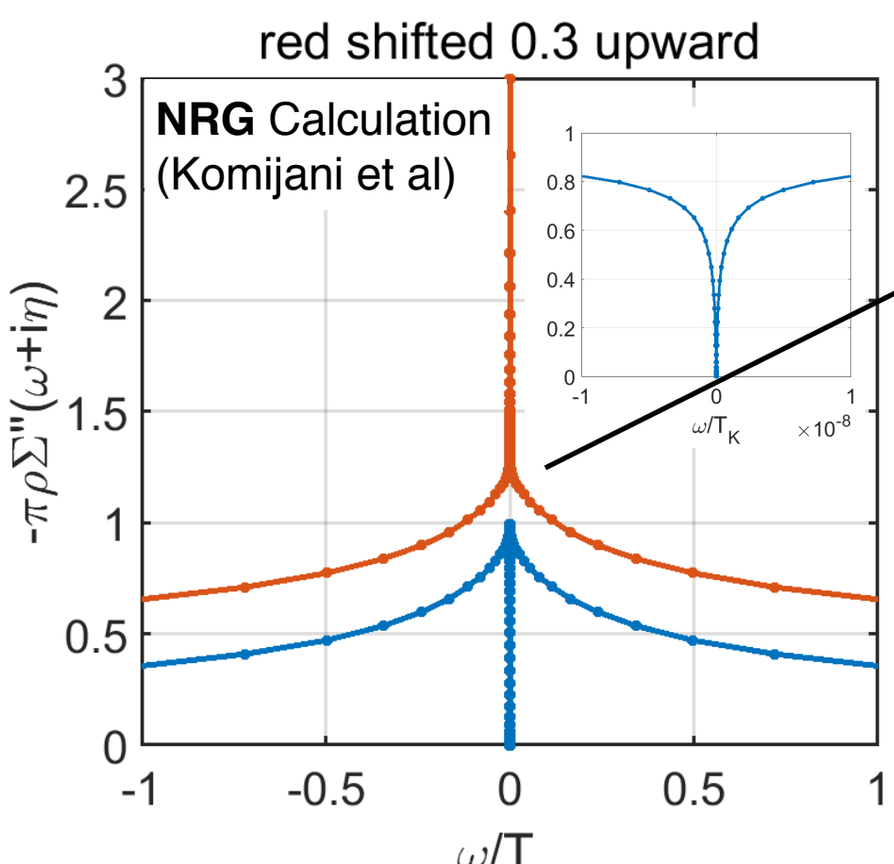
$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\lambda\sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_\lambda(0)\cdot\vec{S} + \delta J[\vec{\sigma}_1(0) - \vec{\sigma}_2(0)]\cdot\vec{S}$$



2-channel Kondo



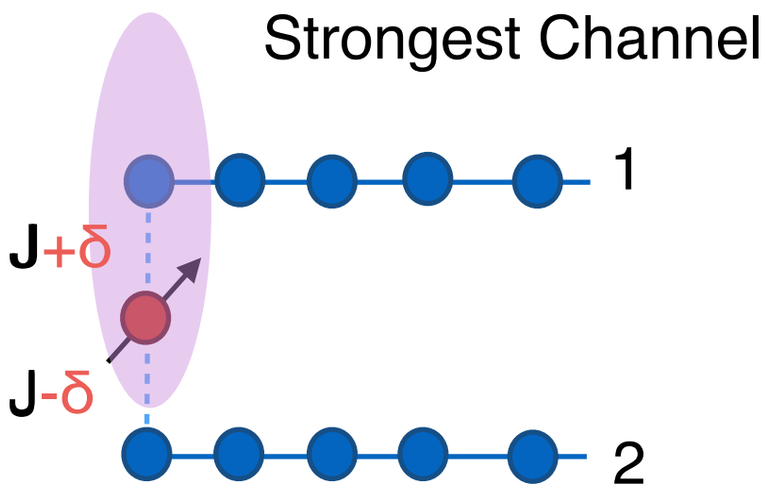
$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$



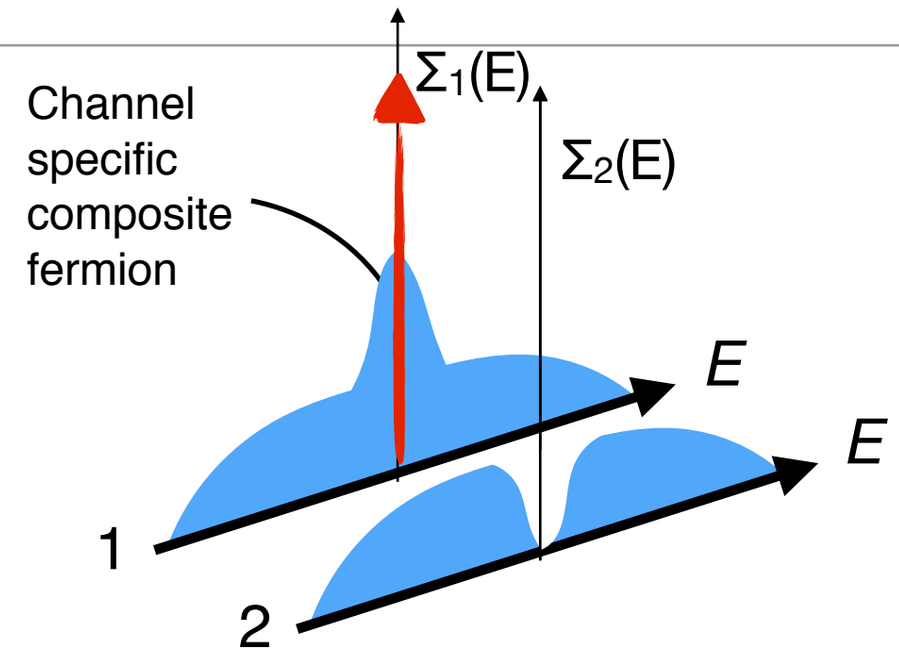
$$\Sigma_{\lambda\lambda'}(\omega - i\delta) = V_\lambda V^*_{\lambda'} \frac{1}{\omega - i\delta}$$

Order Parameter Fractionalization (Induced)

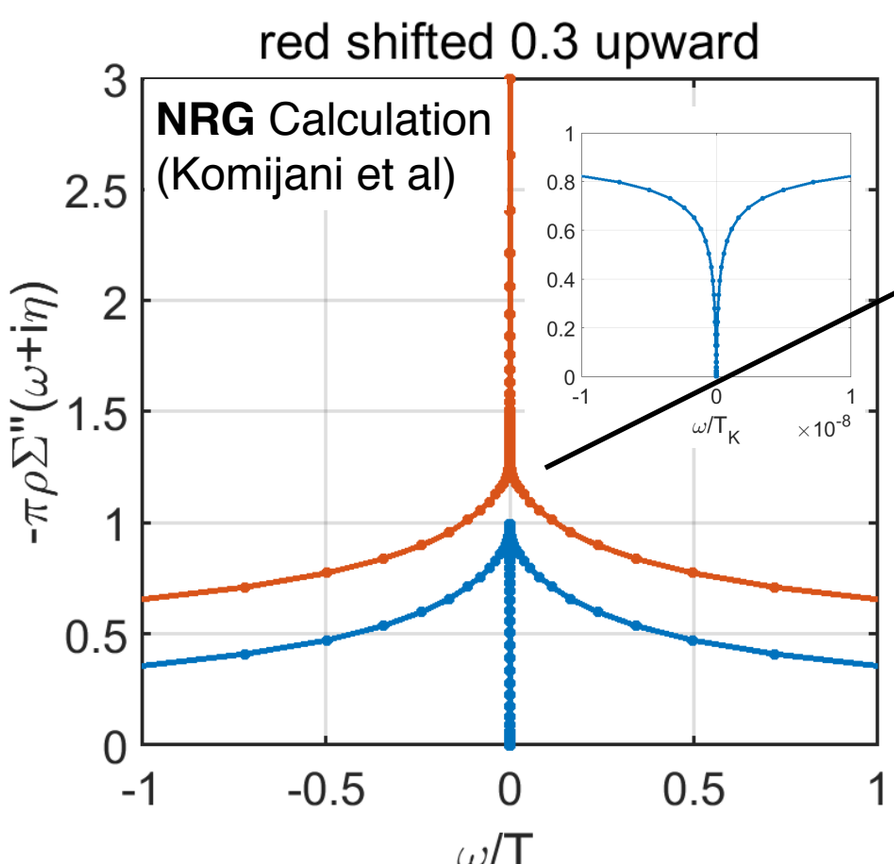
$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\lambda\sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_{\lambda}(0)\cdot\vec{S} + \delta J[\vec{\sigma}_1(0) - \vec{\sigma}_2(0)]\cdot\vec{S}$$



2-channel Kondo



$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$



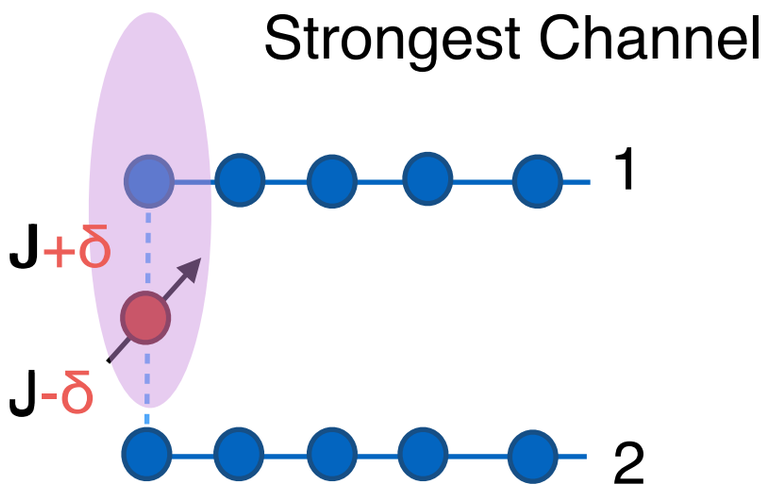
$$\Sigma_{\lambda\lambda'}(\omega - i\delta) = V_\lambda V^*_{\lambda'} \frac{1}{\omega - i\delta}$$

$$\Sigma_{\lambda\lambda'}(2,1) \xrightarrow{|t_2 - t_1| \to \infty} V_\lambda(2) V_{\lambda'}(1) \mathrm{sgn}(t_2 - t_1)$$

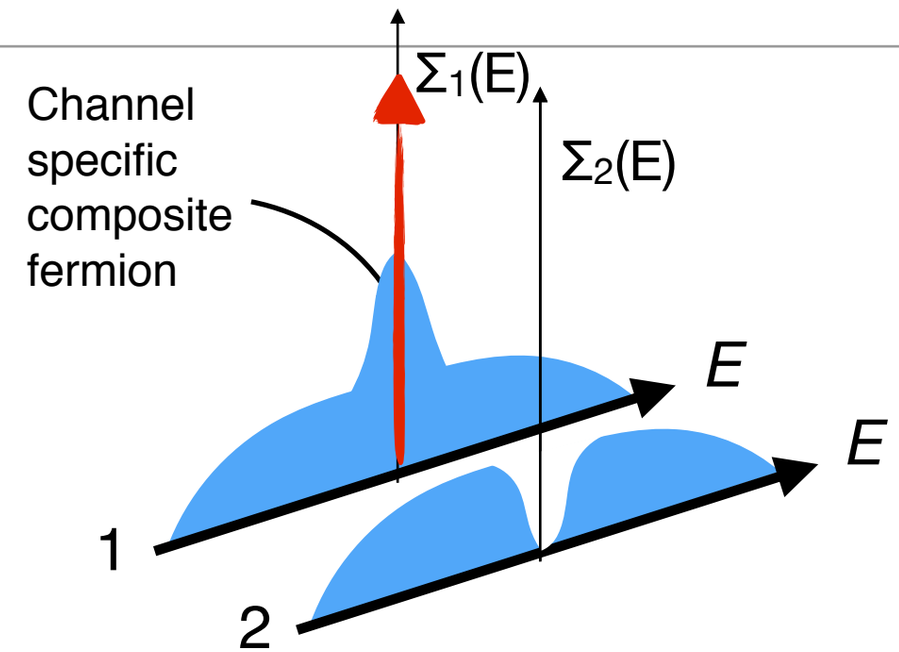
ODLRO in Time

Order Parameter Fractionalization (Induced)

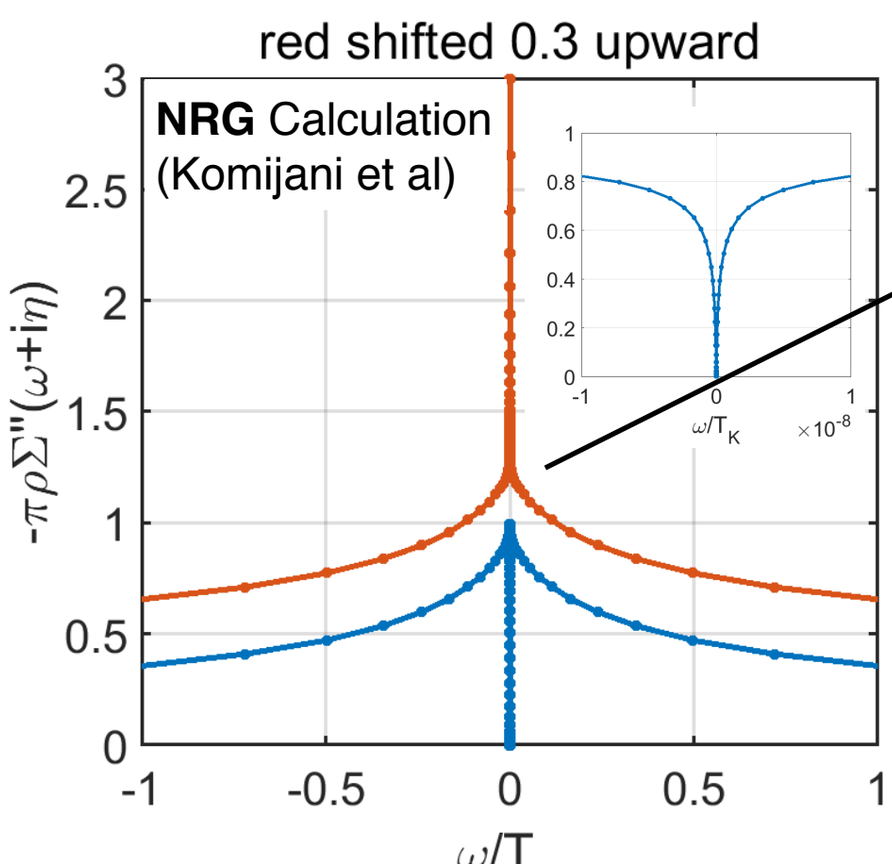
$$H = \sum_{\vec{k}\sigma\lambda} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\lambda\sigma} c_{\vec{k}\lambda\sigma} + J \sum_{\lambda=1,2} \vec{\sigma}_\lambda(0)\cdot\vec{S} + \delta J[\vec{\sigma}_1(0) - \vec{\sigma}_2(0)]\cdot\vec{S}$$



2-channel Kondo



$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$



$$(\vec{\sigma}_{\alpha\beta}\cdot\vec{S})\psi_{\lambda\beta} \rightarrow V_\lambda f_\alpha(0)$$

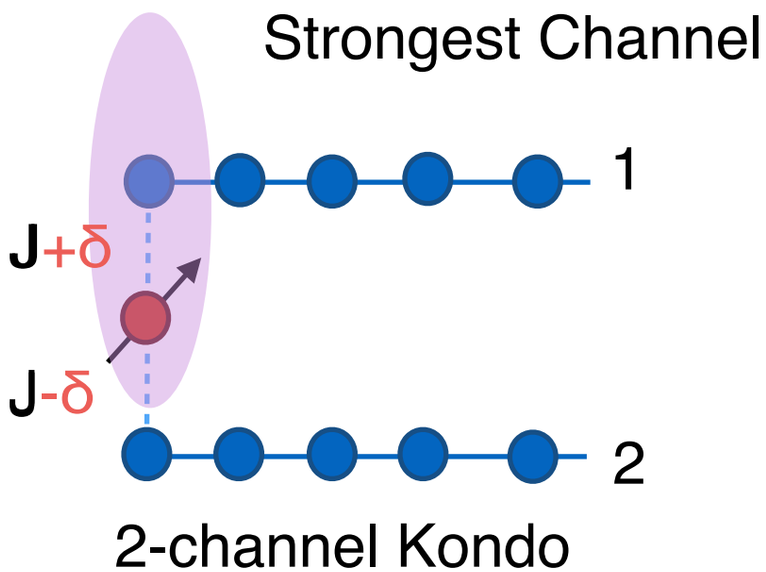
Fractionalization of Composite into Fermion+OP

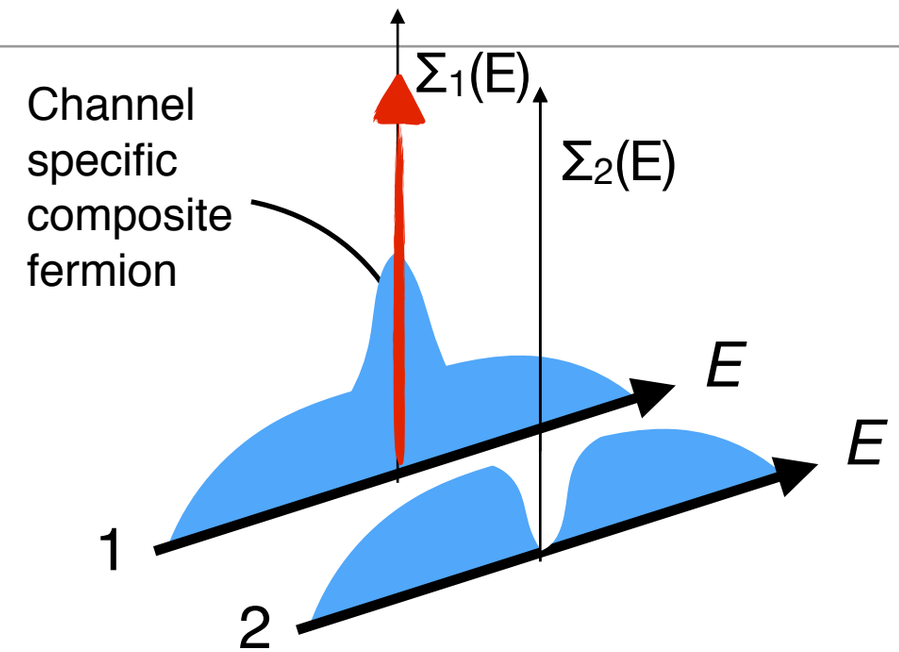
$$\Sigma_{\lambda\lambda'}(2,1) \xrightarrow{|t_2-t_1|\to\infty} V_\lambda(2)V_{\lambda'}(1)\mathrm{sgn}(t_2-t_1)$$

ODLRO in Time

Spontaneous Order Fractionalization

Order Fractionalization (Spontaneous)

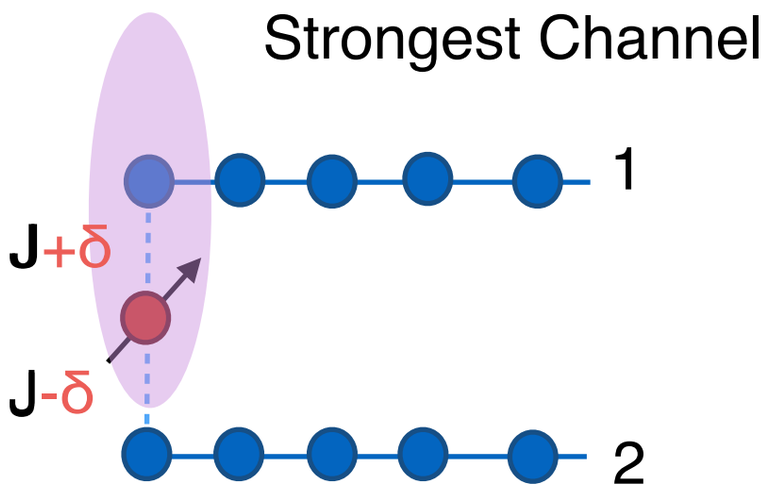




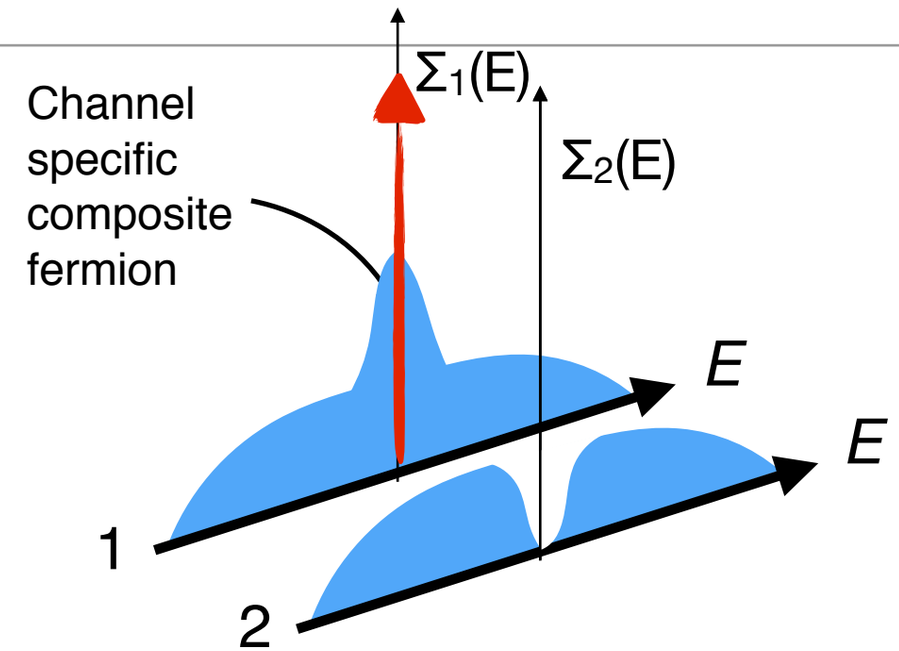
$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$

$$\boxed{(\vec{\sigma}_{\alpha\beta} \cdot \vec{S}\ )\psi_{\lambda\beta} \rightarrow V_\lambda f_\alpha(0)}$$

Order Fractionalization (Spontaneous)

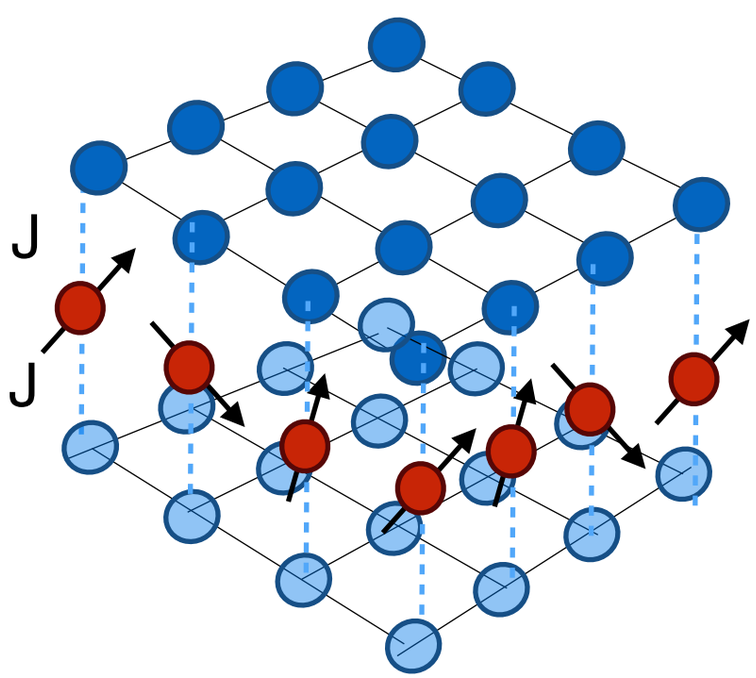


2-channel Kondo



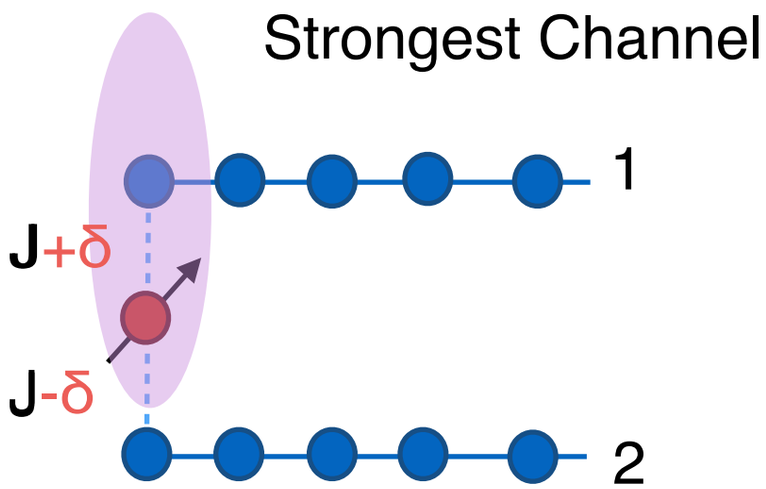
$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$

$$\boxed{(\vec{\sigma}_{\alpha\beta} \cdot \vec{S}\ )\psi_{\lambda\beta} \rightarrow V_\lambda f_\alpha(0)}$$

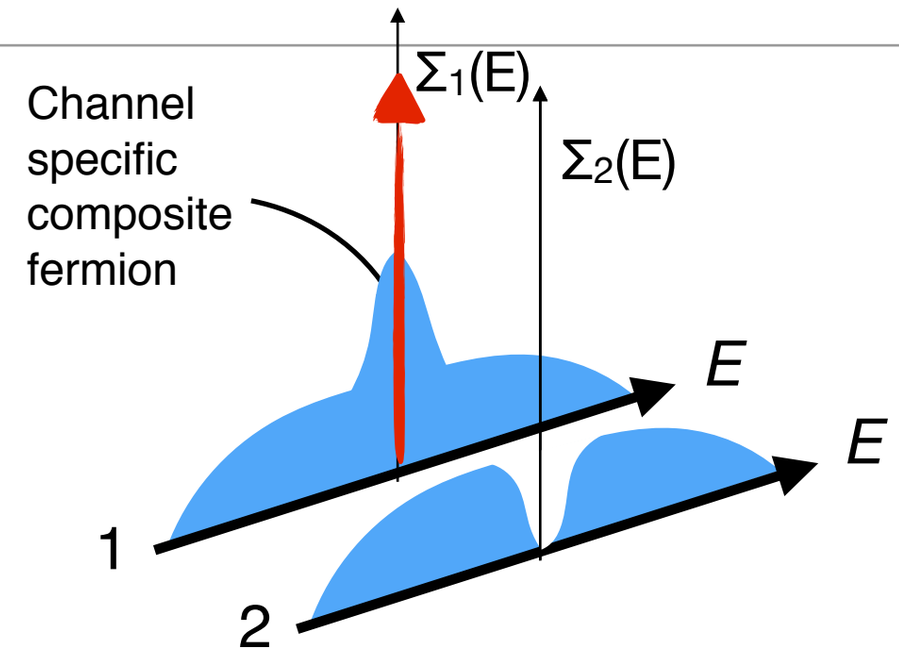


2-channel Kondo Lattice

Order Fractionalization (Spontaneous)



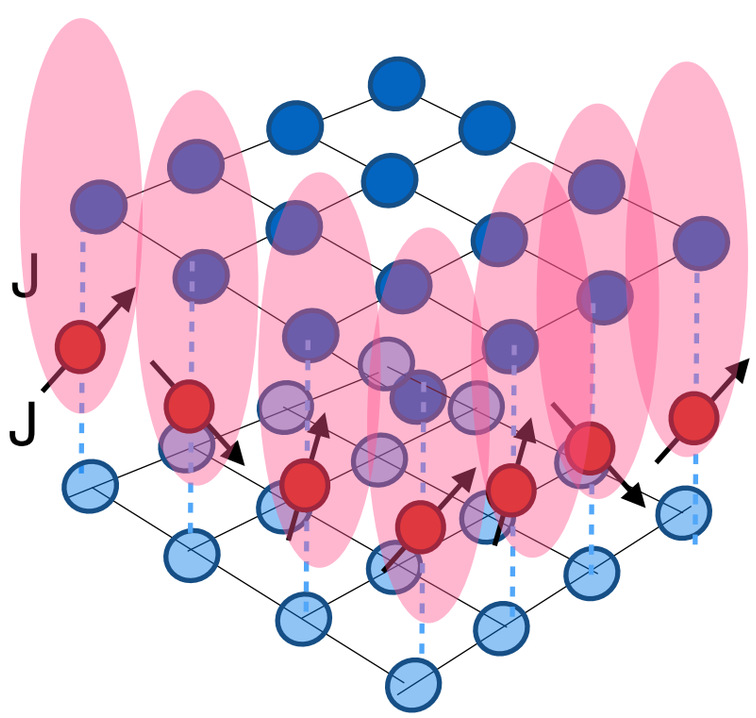
2-channel Kondo



$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$

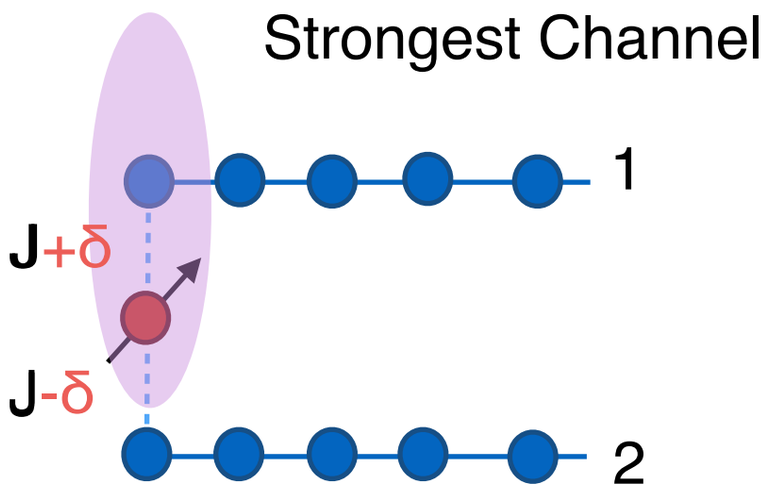
$$\boxed{(\vec{\sigma}_{\alpha\beta} \cdot \vec{S}\ )\psi_{\lambda\beta} \rightarrow V_\lambda f_\alpha(0)}$$

Spinor OP Forms Spontaneously

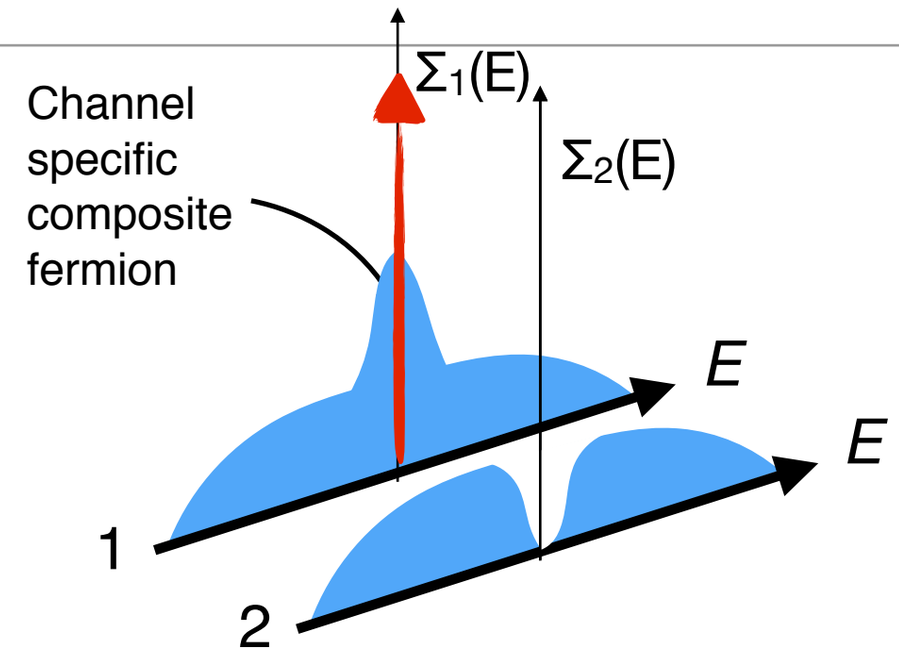


2-channel Kondo Lattice

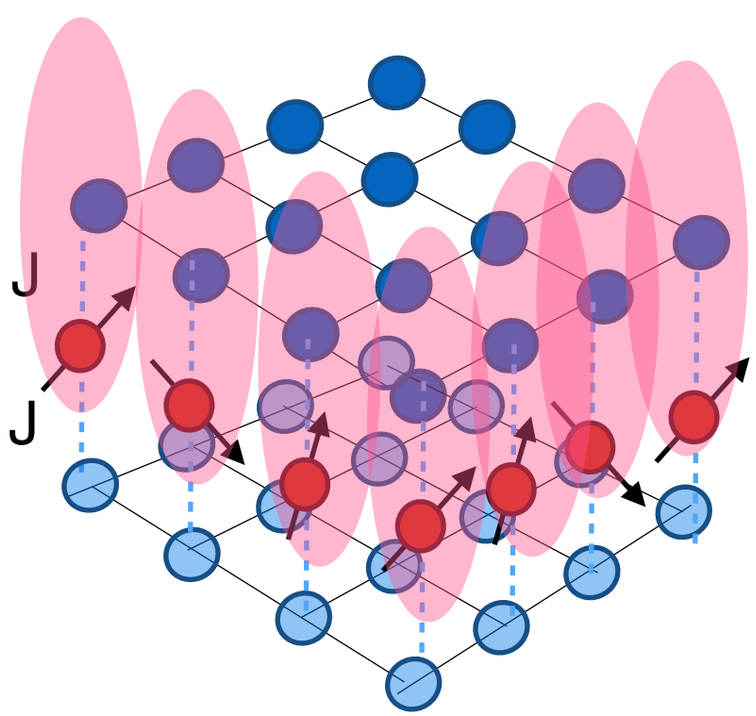
Order Fractionalization (Spontaneous)



2-channel Kondo



$$V_\lambda = \begin{pmatrix} V \\ 0 \end{pmatrix}$$



2-channel Kondo Lattice

$$\boxed{(\vec{\sigma}_{\alpha\beta} \cdot \vec{S}\ )\psi_{\lambda\beta} \rightarrow V_\lambda f_\alpha(0)}$$

Spinor OP Forms Spontaneously

$$\Sigma_{\lambda\lambda'}(2,1) \xrightarrow{|2-1|\rightarrow\infty} V_\lambda(2)V_{\lambda'}(1)g(2-1)$$

ODLRO in Space Time

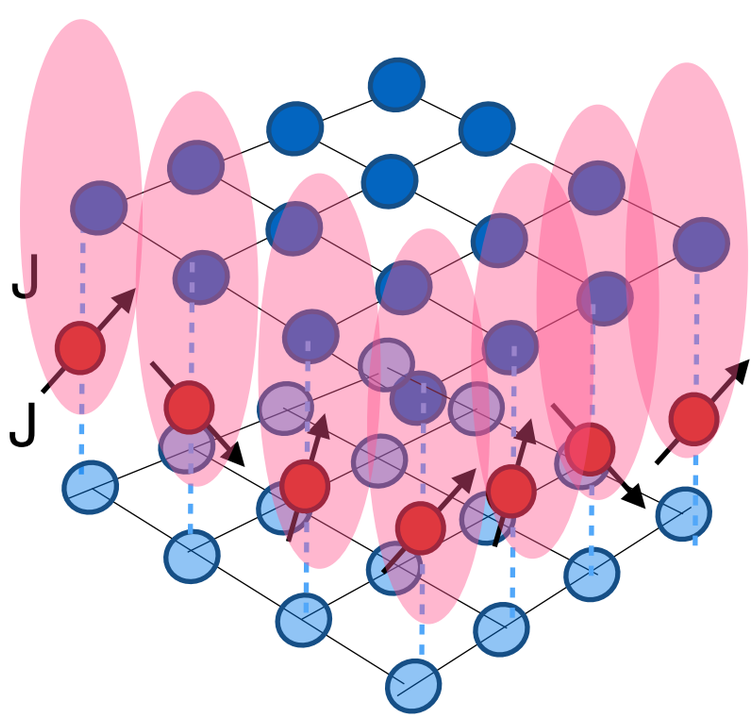
Order Parameter Fractionalization Hypothesis

P. Chandra, P. Coleman, Y. Komijani

Composite order Fractionalized

$$\Psi = \langle \left(\psi_1^\dagger \vec{\sigma} \psi_1 - \psi_2^\dagger \vec{\sigma} \psi_2 \right) \cdot \vec{S} \rangle$$
$$\propto |V_1|^2 - |V_2|^2$$

cf Emery and Kivelson 1993



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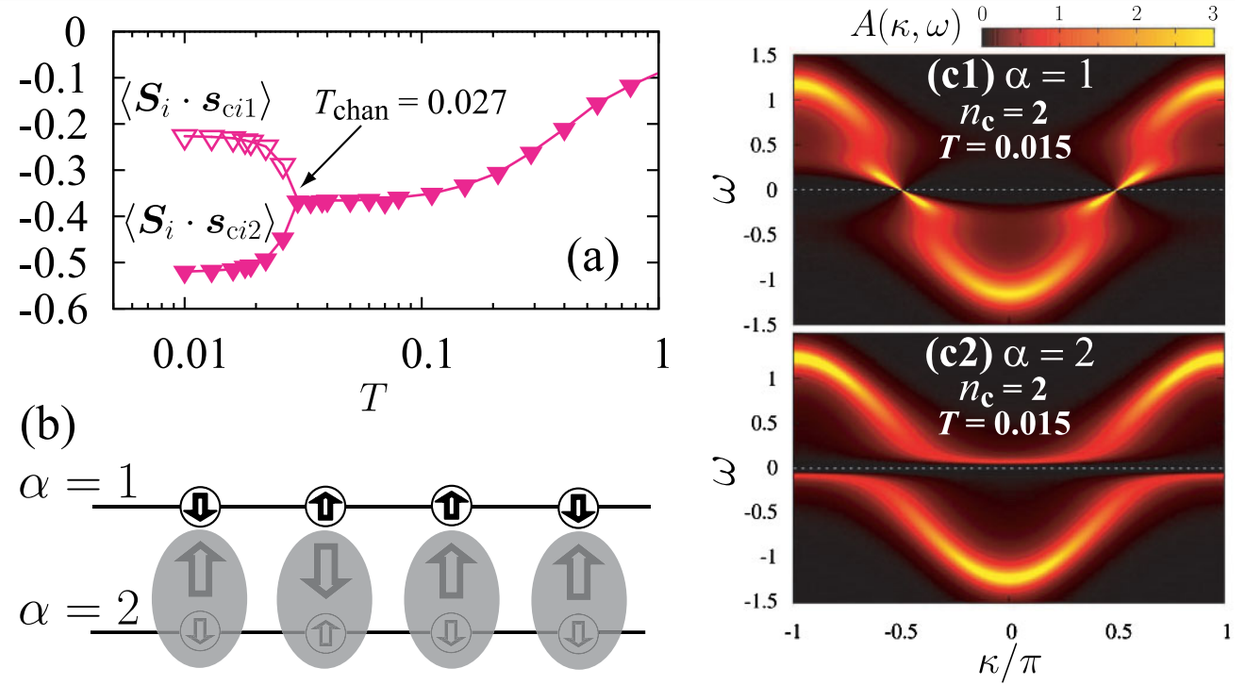
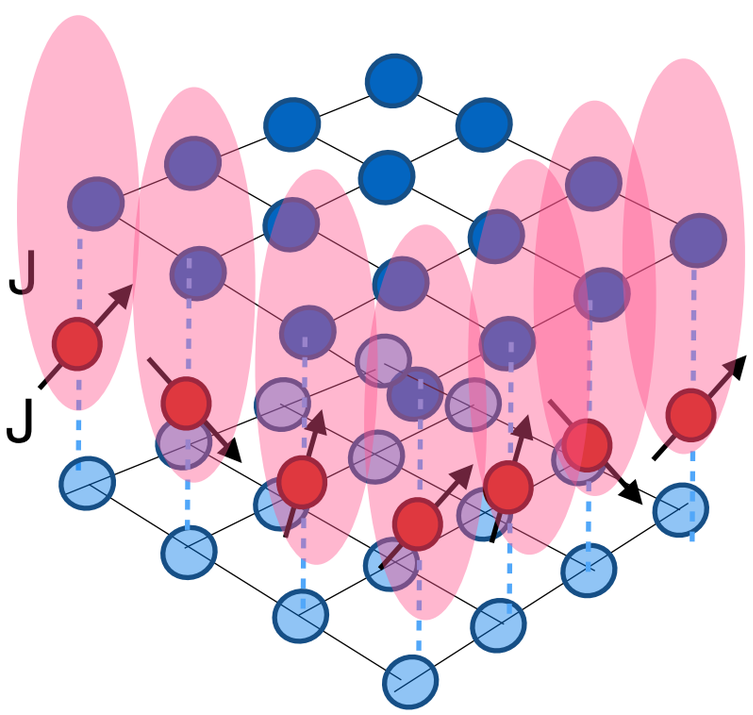


Hoshino, Otsuki & Kuromoto, PRL 107, 247202 (2011)



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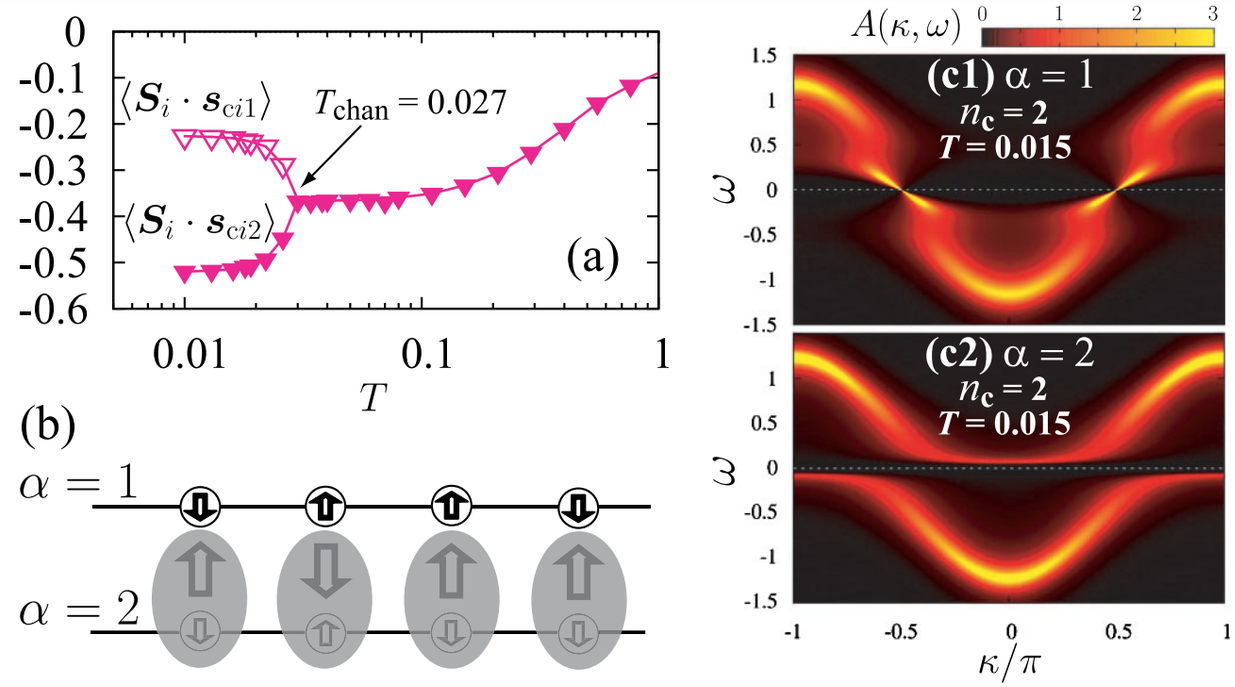

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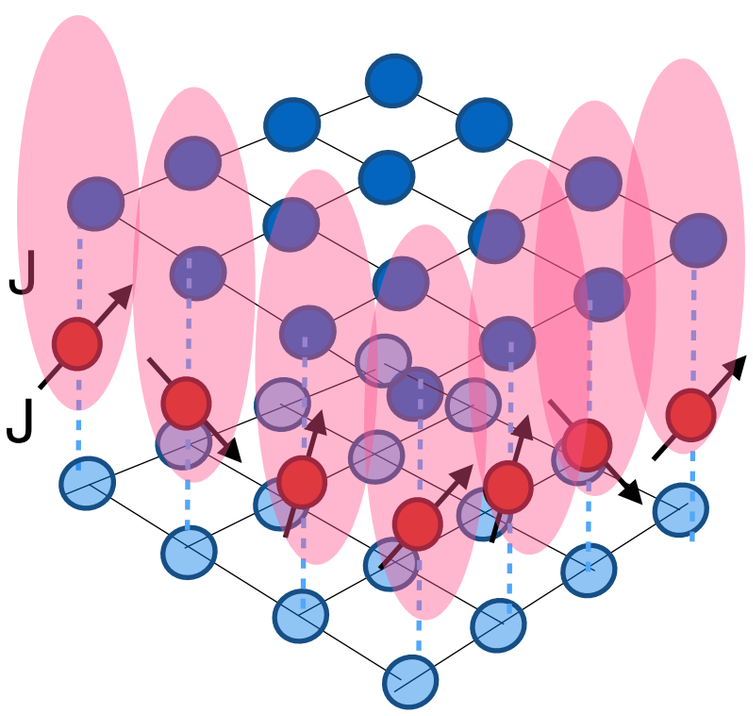

2-channel Kondo Lattice

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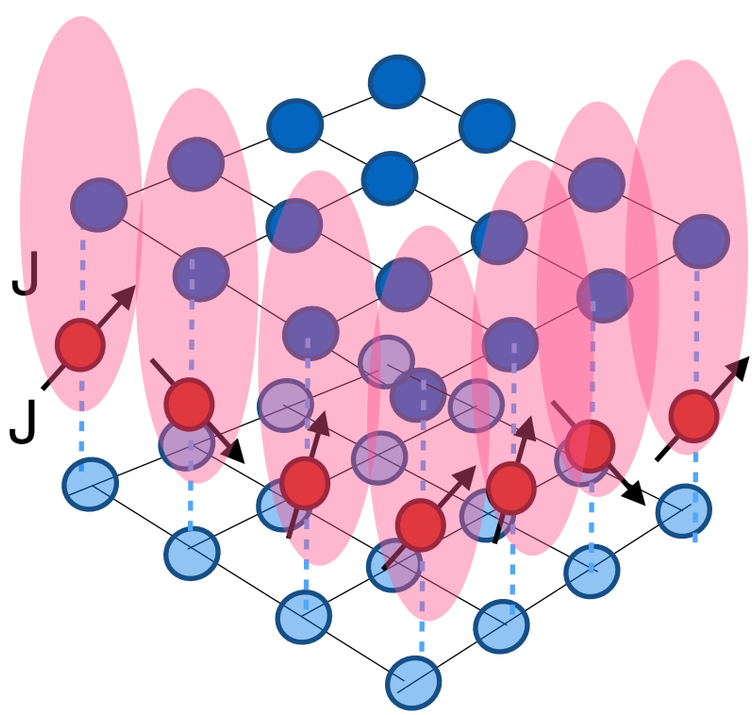
ODLRO in Space Time

Order Parameter Fractionalization Hypothesis

P. Chandra, PC, Y. Komijani, A. Toth

Composite Order

2-channel	$(\vec{S}\cdot\vec{\sigma})_{\alpha\beta}\psi_{\lambda\beta}$	$V_\lambda f_\alpha$	Composite Multipole	$\langle\left(\psi_1^\dagger\vec{\sigma}\psi_1-\psi_2^\dagger\vec{\sigma}\psi_2\right)\cdot\vec{S}\rangle$



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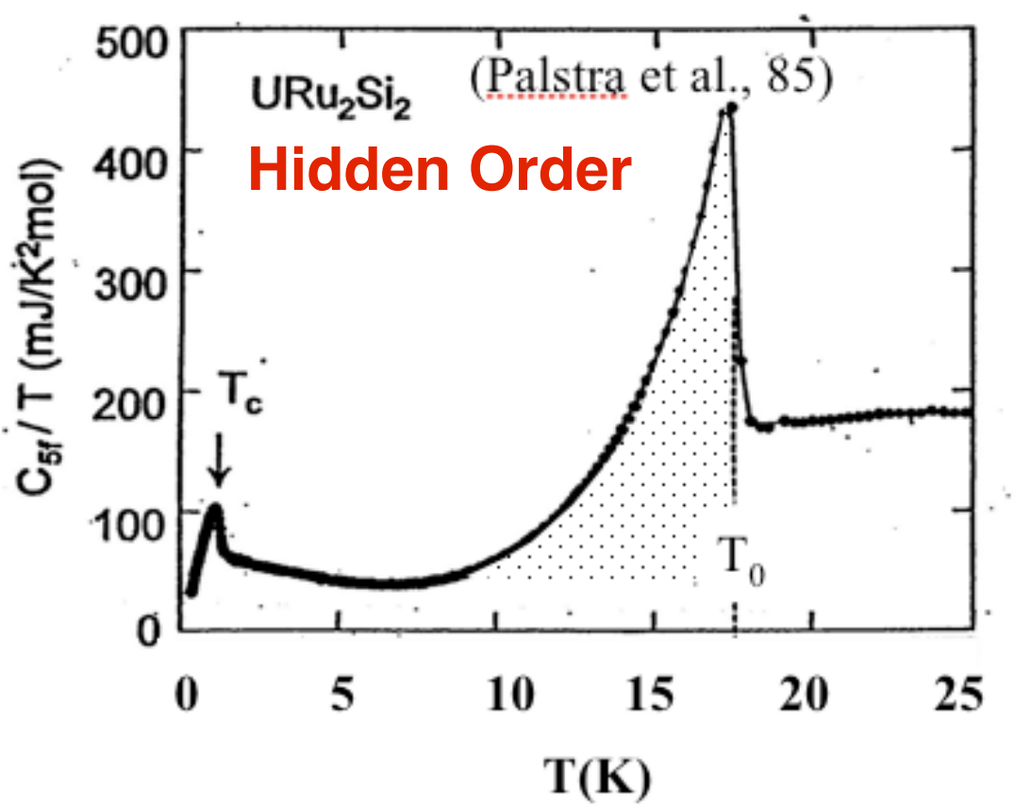

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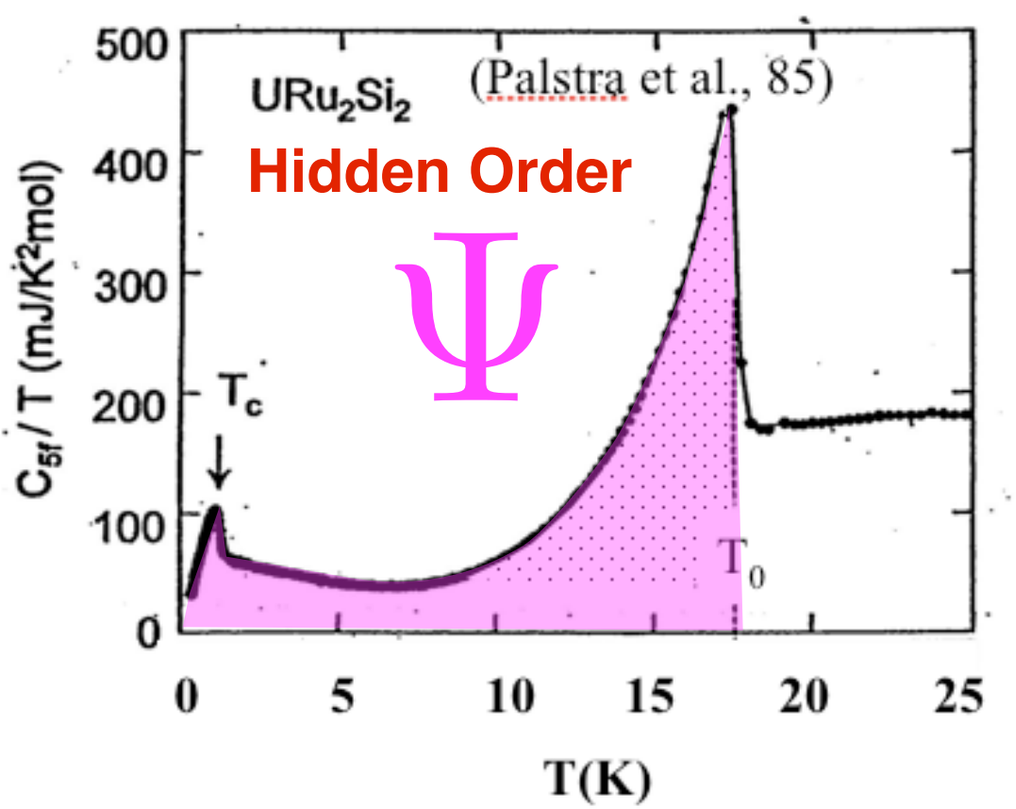

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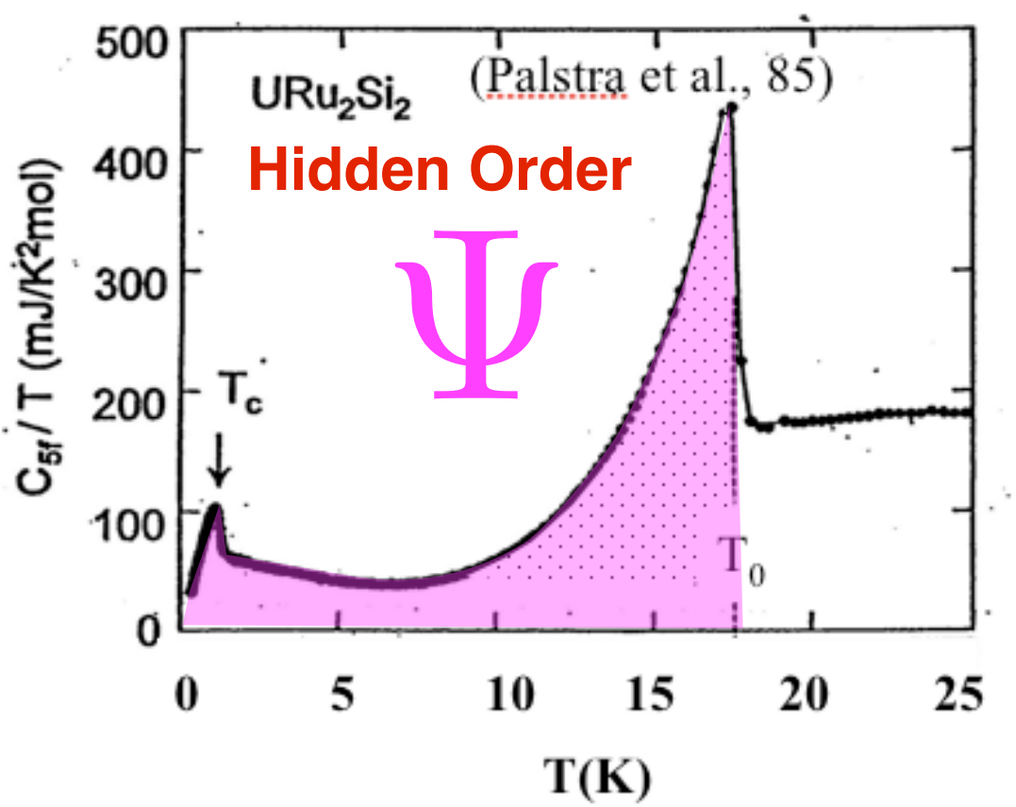

See eg:

J=3 Kiss & Fazekas, Phys Rev B, (2005)
J=4 Kotliar & Haule, Nat Phys, (2009)
J=5 Ikeda et al, Nat. Phys (2012)

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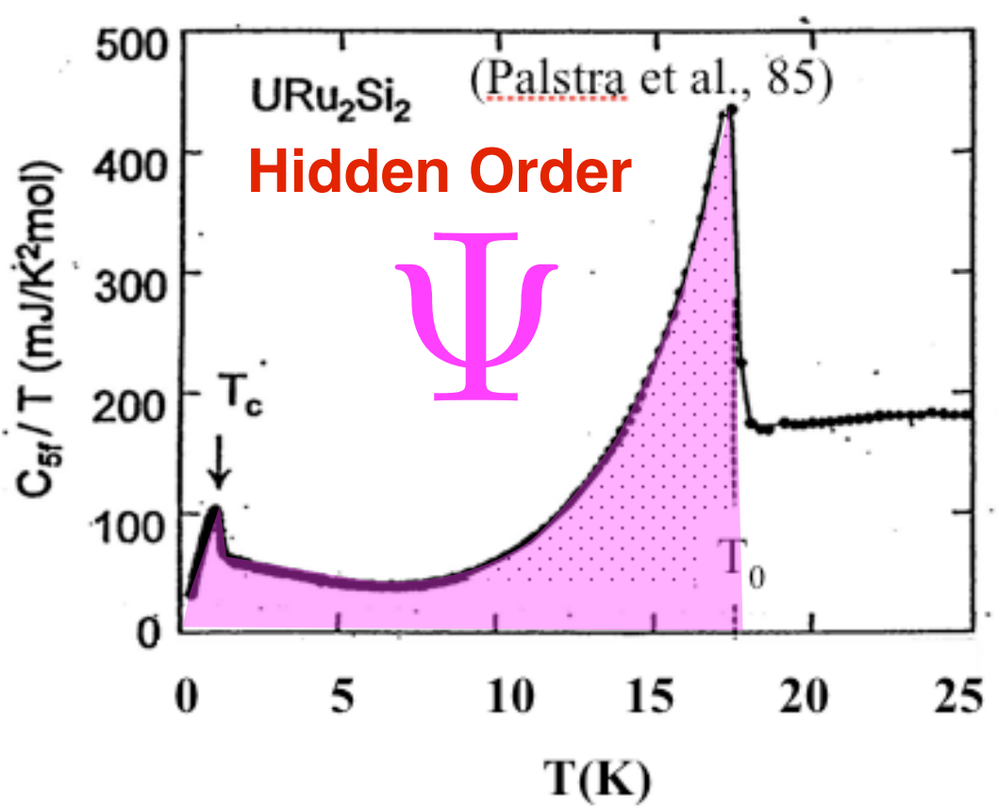
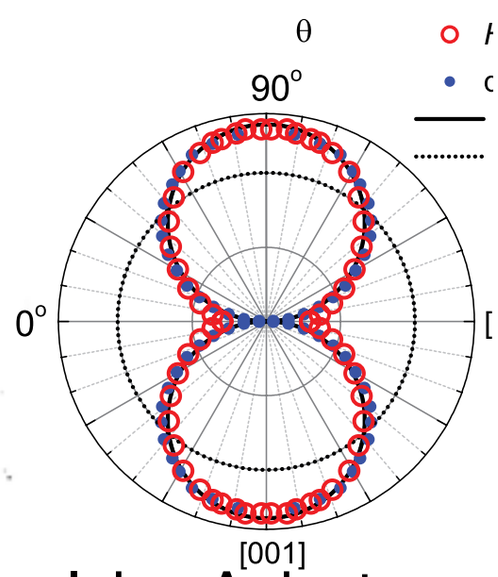
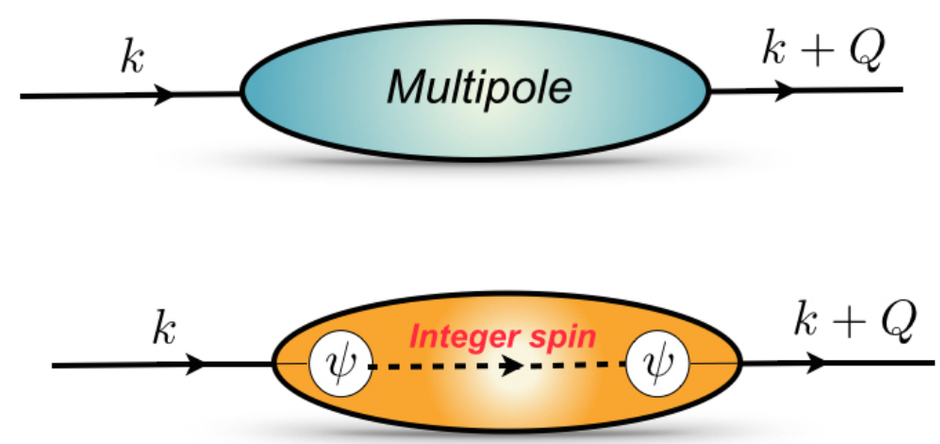




Ising Anisotropy



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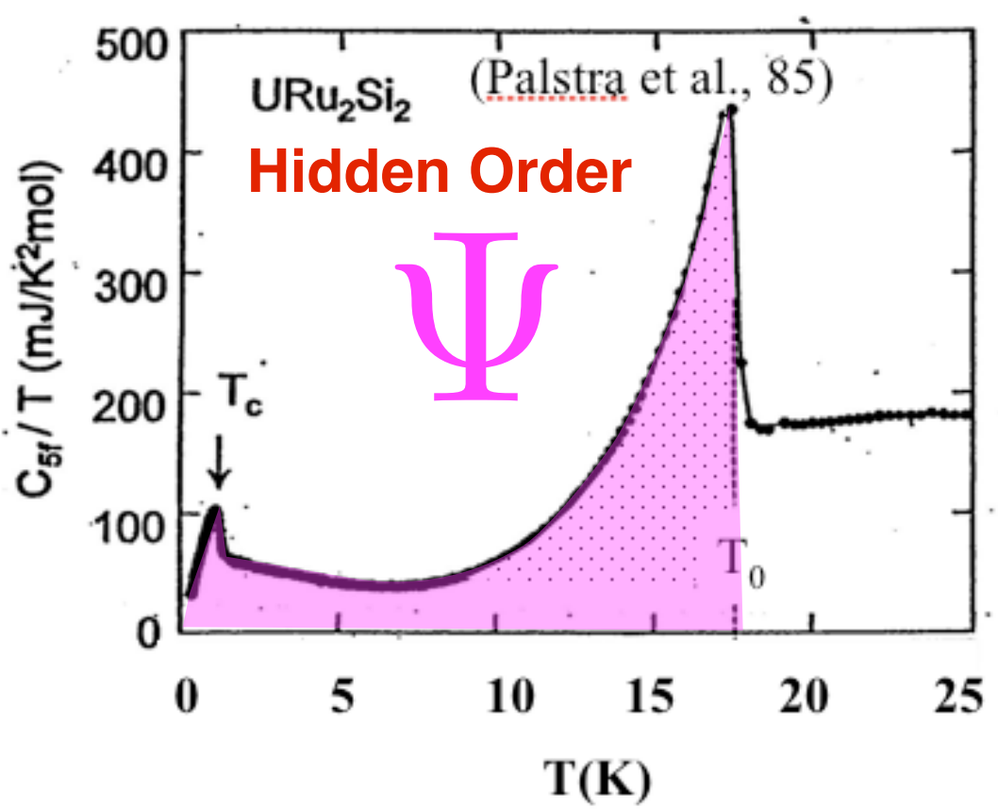

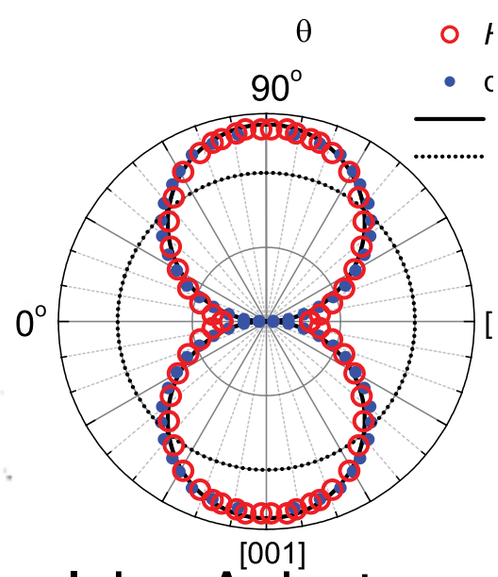

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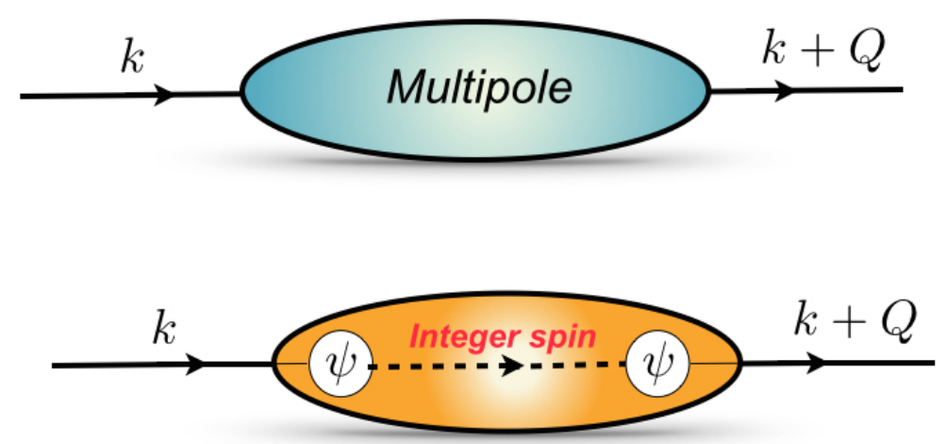

$$\psi = \sqrt{\text{Multipole}} = \text{Spinorial OP}$$

Chandra, Coleman, Flint, Nature (2013)

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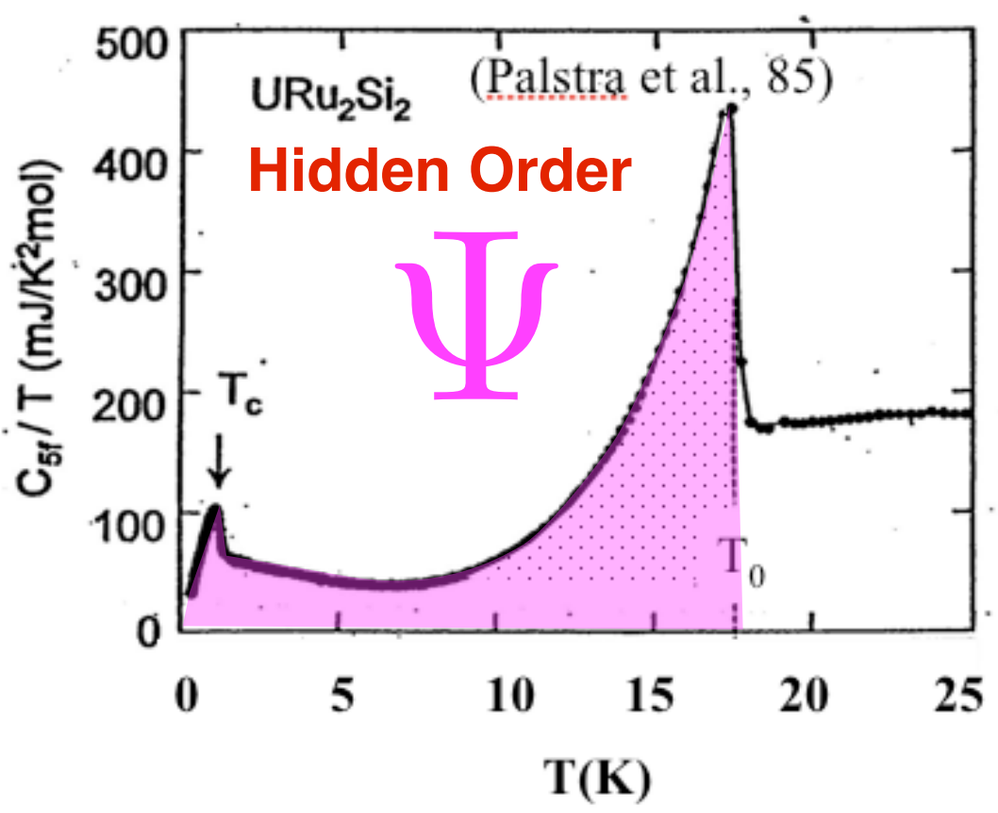
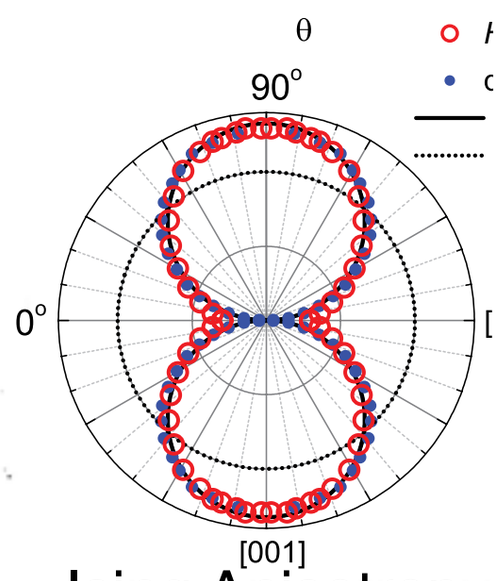
		$\Psi_\alpha\hat{\chi}\lambda$	Hastatic	$\langle c^\dagger\vec{\sigma}(\vec{I}\cdot\vec{\tau})c\rangle \propto \Psi^\dagger\vec{\sigma}\,\Psi$





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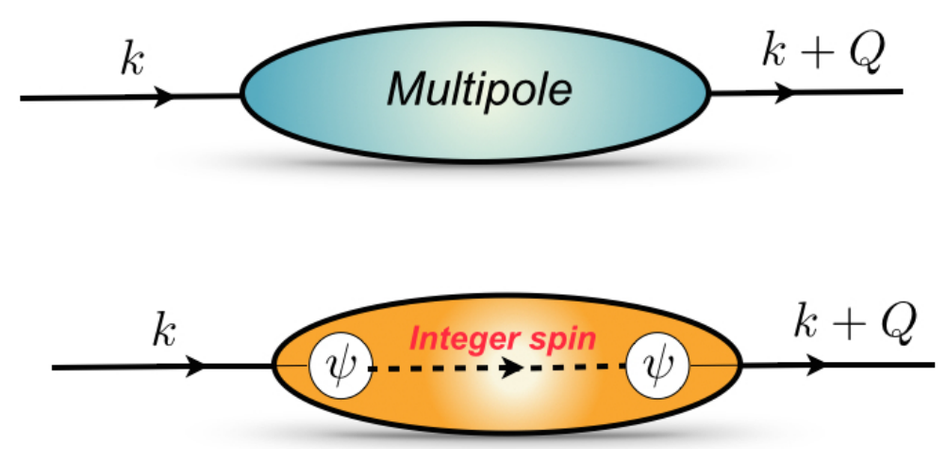

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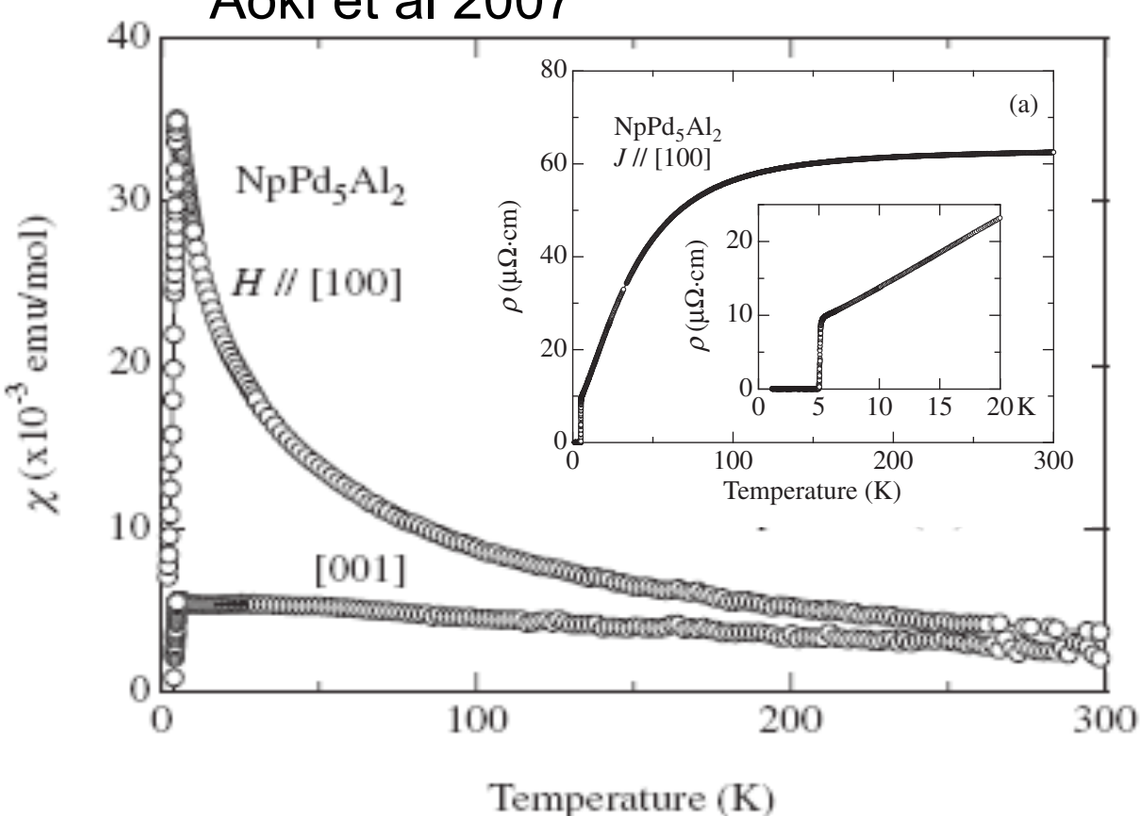
Aoki et al 2007



See: Flint, Dzero, PC, Nat Phys. (2008)

$NpPd_5Al_2$ T_C = 4.5K

Curie Law SC

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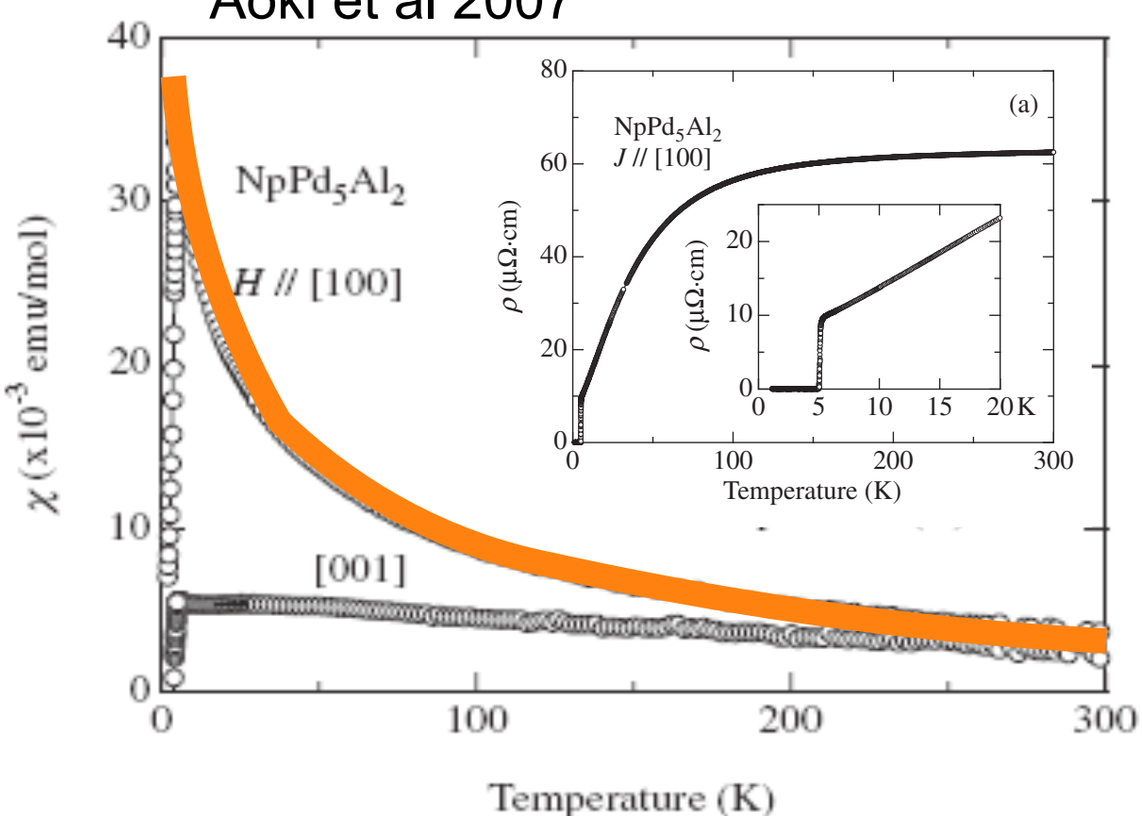

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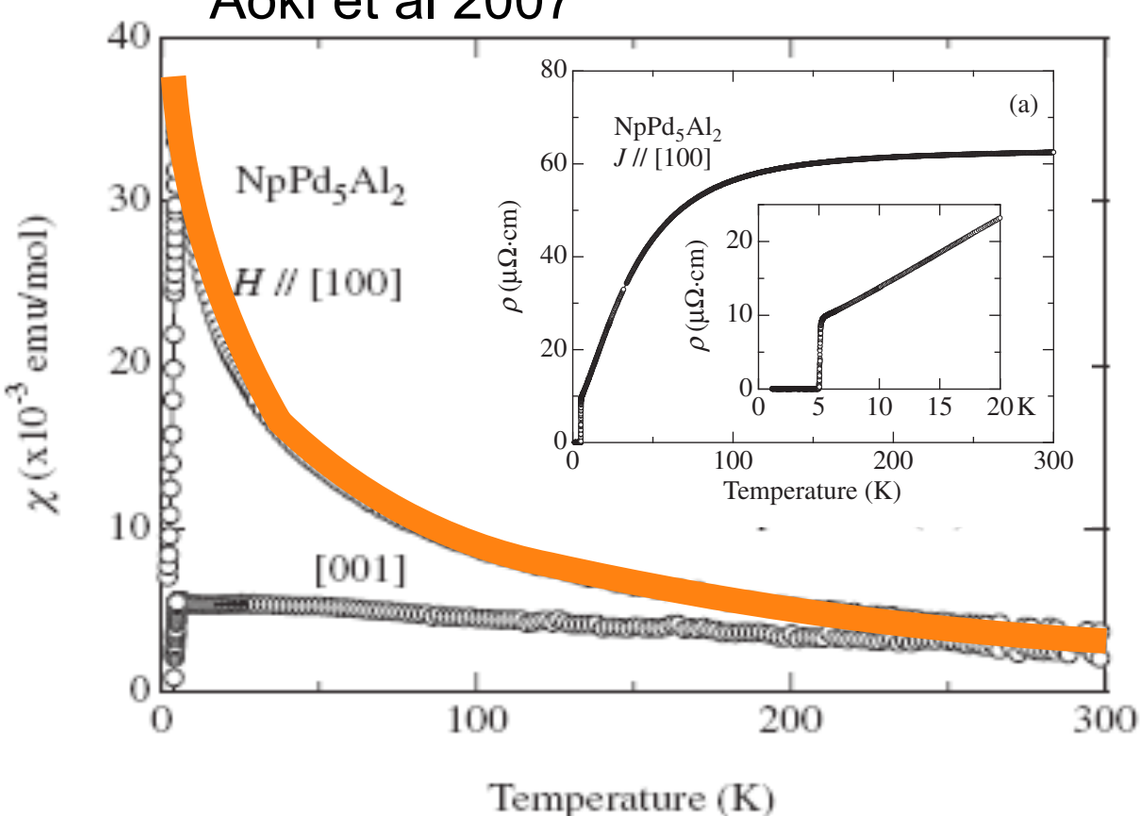
Aoki et al 2007



Composite order

$$\langle(\psi_1\vec{\sigma}\sigma_2\psi_2)\cdot\vec{S}\rangle \propto (V_1\Delta_2 - V_2\Delta_1)$$

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$NpPd_5Al_2$ $T_C = 4.5K$

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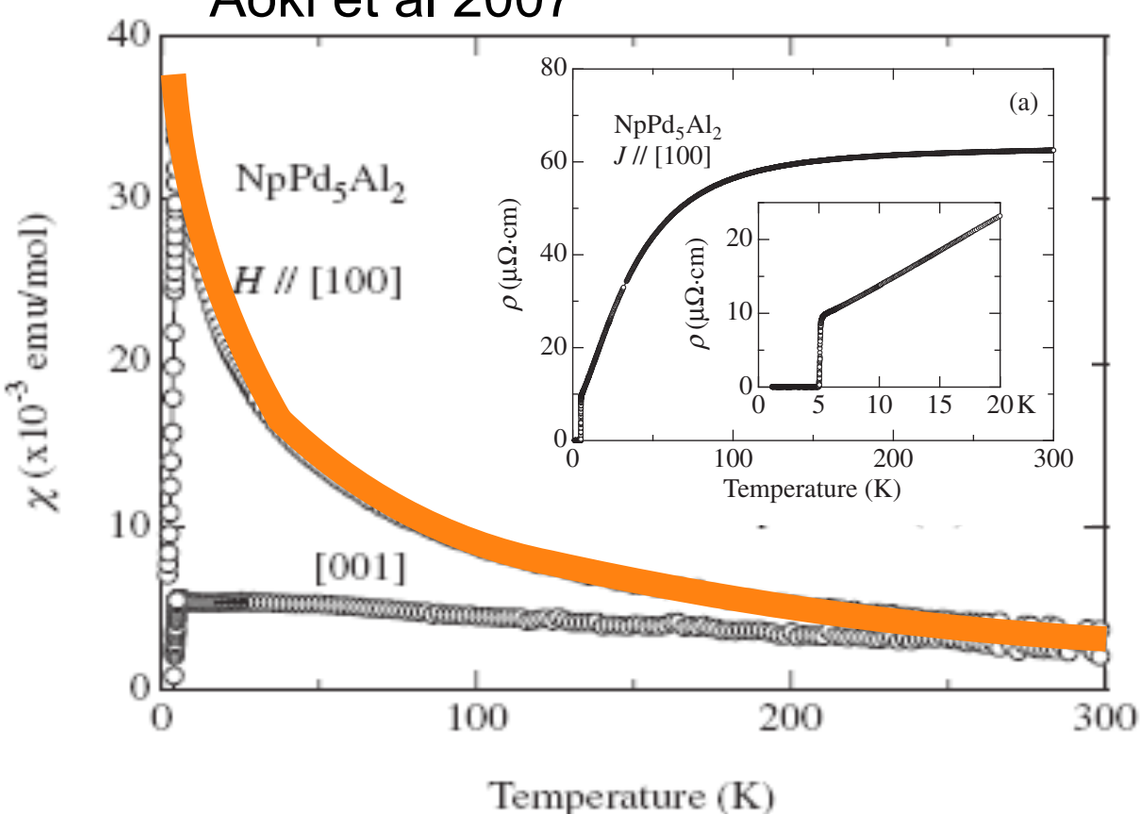
		$\Psi_\alpha\hat{\chi}\lambda$	Hastatic	$\langle c^\dagger\vec{\sigma}(\vec{I}\cdot\vec{\tau})c\rangle \propto \Psi^\dagger\vec{\sigma}\Psi$

Aoki et al 2007



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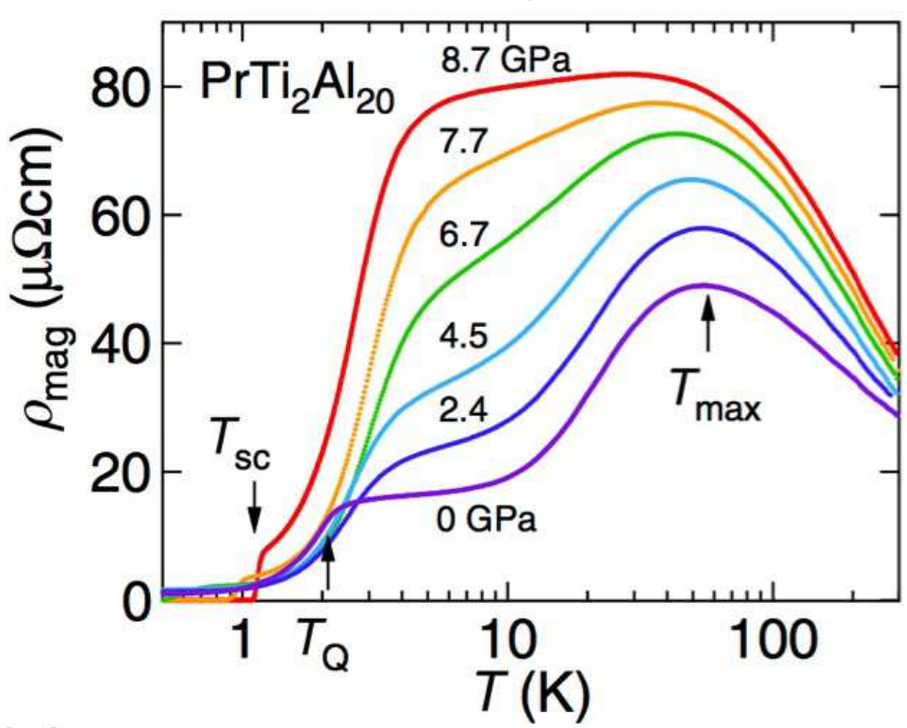
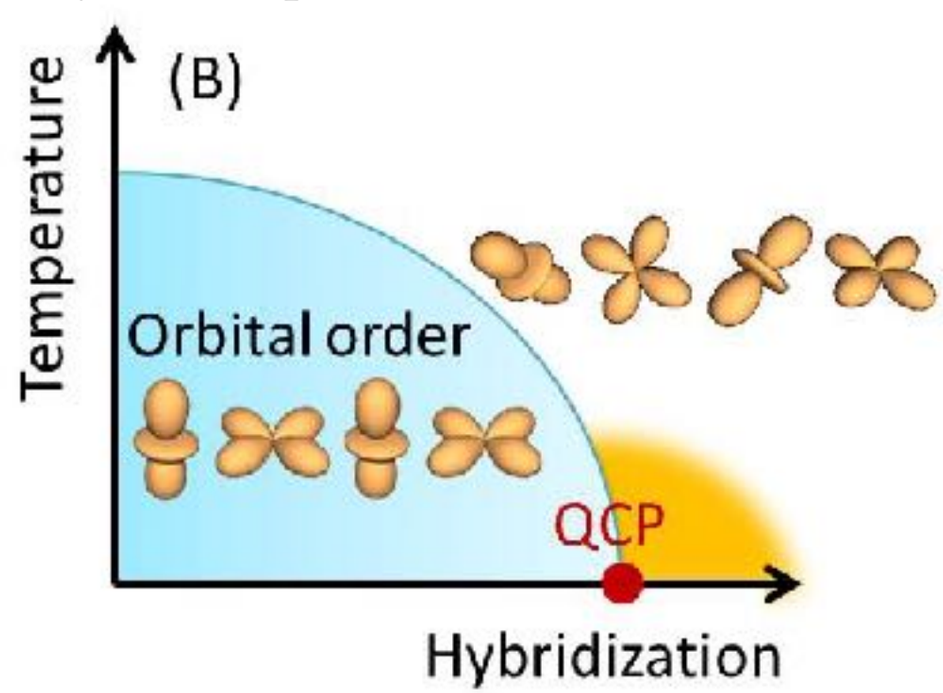
A. Sakai, K. Kuga, and S. Nakatsuji, J. Phys. Soc. Jpn. **81**, 083702 (2012).





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Kondo	$(\vec{S}\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta$	$V f_\alpha$	HF	$\langle\psi^\dagger(\vec{\sigma}\cdot\vec{S})\psi\rangle \propto \lvert V\rvert^2$
Majorana		$(\vec{\sigma}\cdot\vec{\eta})_{\alpha\beta}\mathcal{V}_\beta$	Odd-w triplet/ Skyrme Insulator	$\langle\psi_\uparrow\psi_\downarrow\vec{S}\rangle \propto \mathcal{V}^T\vec{\sigma}\sigma_2\mathcal{V}$
2-channel	$(\vec{S}\cdot\vec{\sigma})_{\alpha\beta}\psi_{\lambda\beta}$	$V_\lambda f_\alpha$	Composite Multipole	$\langle(\psi_1^\dagger\vec{\sigma}\psi_1 - \psi_2^\dagger\vec{\sigma}\psi_2)\cdot\vec{S}\rangle$
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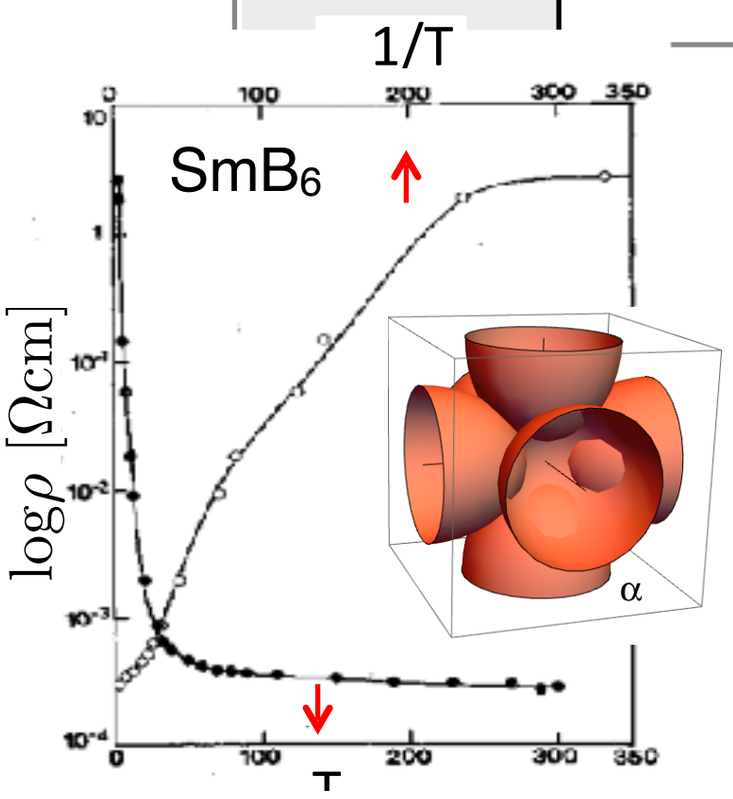

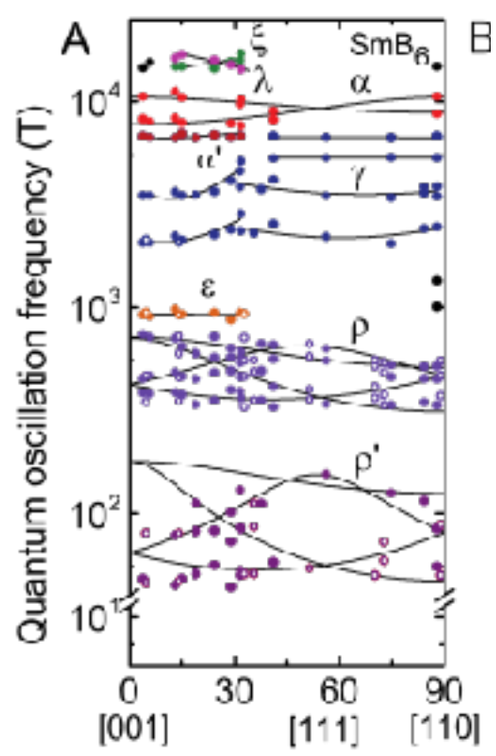

Tan *et al.* Science 349, 287 (2015)

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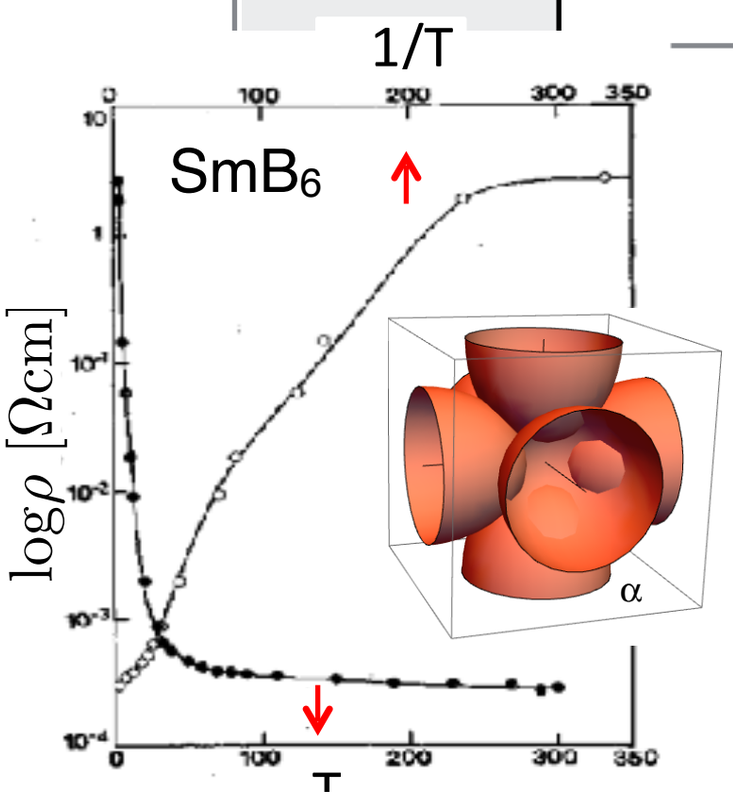
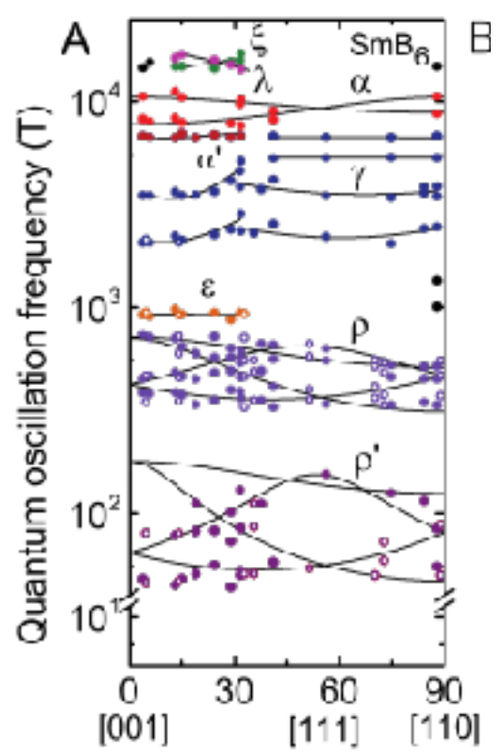
		$V_\lambda f_\alpha + \Delta_\lambda\bar{\alpha} f^\dagger_{-\alpha}$	Composite Pair	$\langle(\psi_1\vec{\sigma}\sigma_2\psi_2)\cdot\vec{S}\rangle$
		$\Psi_\alpha\hat{\chi}\lambda$	Hastatic	$\langle c^\dagger\vec{\sigma}(\vec{I}\cdot\vec{\tau})c\rangle \propto \Psi^\dagger\vec{\sigma}\Psi$





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Majorana Fractionalization
(cf Kitaev)

Tan *et al.* Science 349, 287 (2015)

Order Parameter Fractionalization Hypothesis

P. Chandra, PC, Y. Komijani

$$(\psi\psi\psi)_\Lambda(x) = V^\lambda_{\alpha\alpha'}(x) f_{\alpha'}(x)$$

Composite Order

Kondo	$(\vec{S}\cdot\vec{\sigma})_{\alpha\beta}\psi_\beta$	$V f_\alpha$	HF	$\langle\psi^\dagger(\vec{\sigma}\cdot\vec{S})\psi\rangle \propto \lvert V\rvert^2$
Majorana		$(\vec{\sigma}\cdot\vec{\eta})_{\alpha\beta}\mathcal{V}_\beta$	Odd-w triplet/ Skyrme Insulator	$\langle\psi_\uparrow\psi_\downarrow\vec{S}\rangle \propto \mathcal{V}^T\vec{\sigma}\sigma_2\mathcal{V}$
2-channel	$(\vec{S}\cdot\vec{\sigma})_{\alpha\beta}\psi_{\lambda\beta}$	$V_\lambda f_\alpha$	Composite M	$\ldots\psi_2\big)\cdot\vec{S}\rangle$
		$V_\lambda f_\alpha + \Delta_\lambda\bar{\alpha} f^\dagger_{-\alpha}$	C F	$)\cdot\vec{S}\rangle$
		$\Psi_\alpha\hat{\chi}\lambda$	H	$\Psi^\dagger\vec{\sigma}\Psi$

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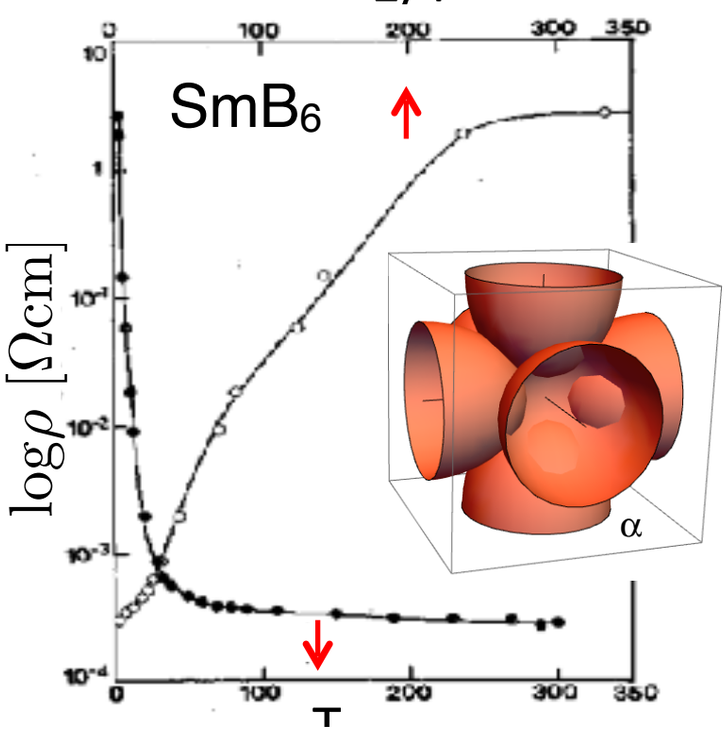
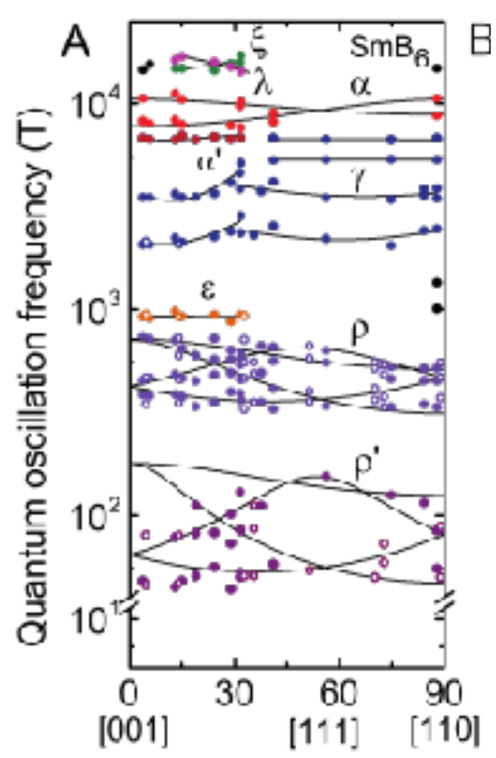
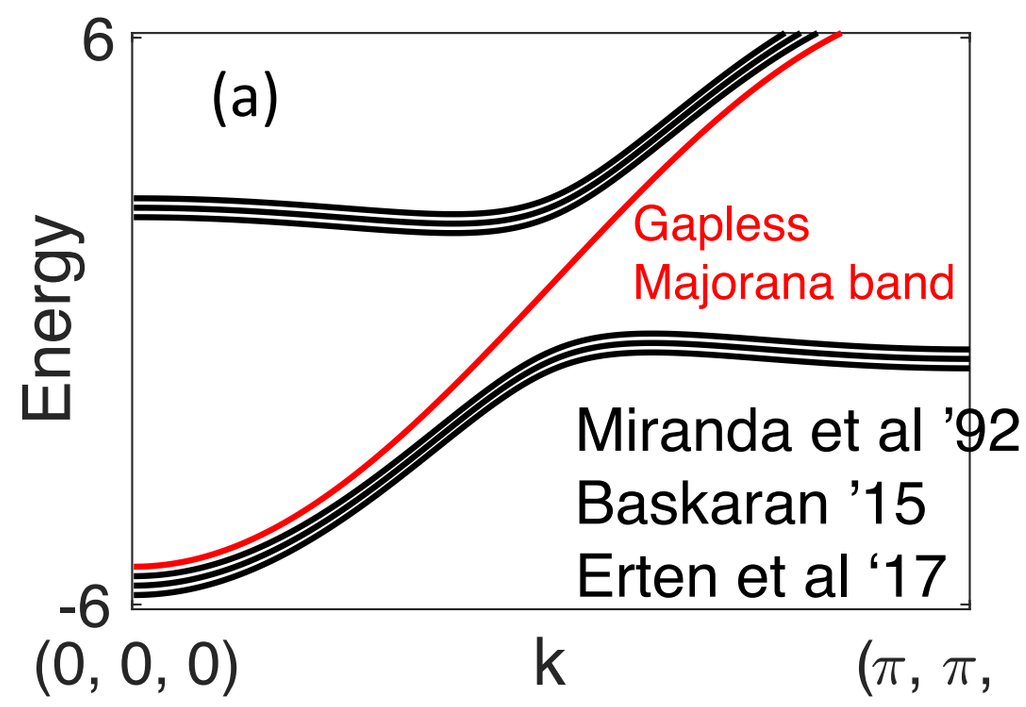




Tan *et al.* Science 349, 287 (2015)



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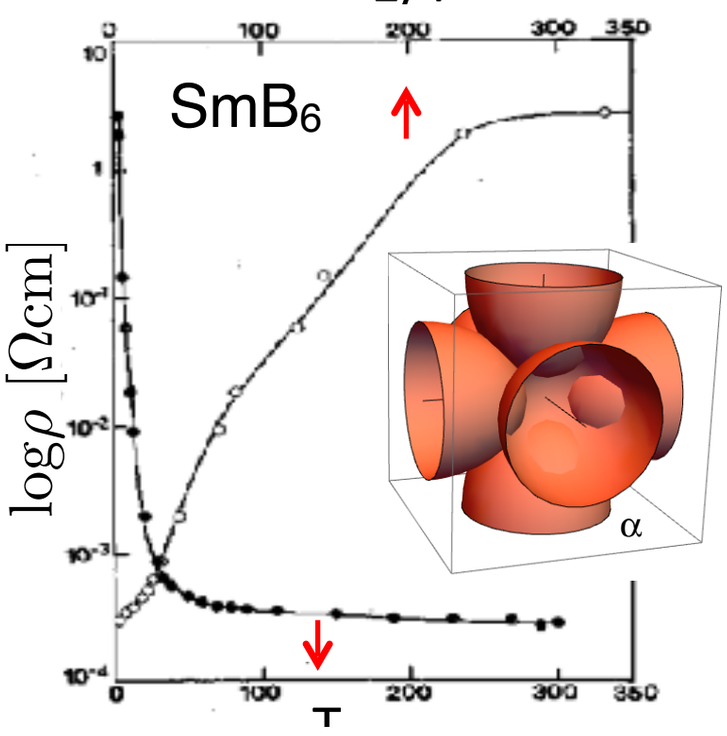

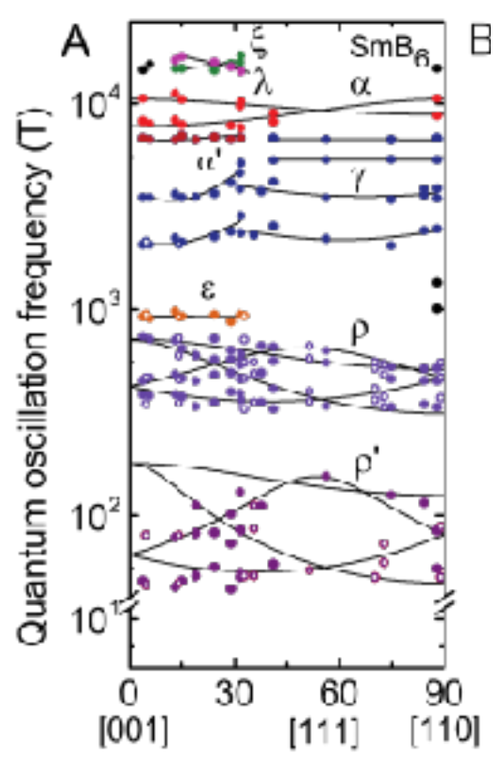

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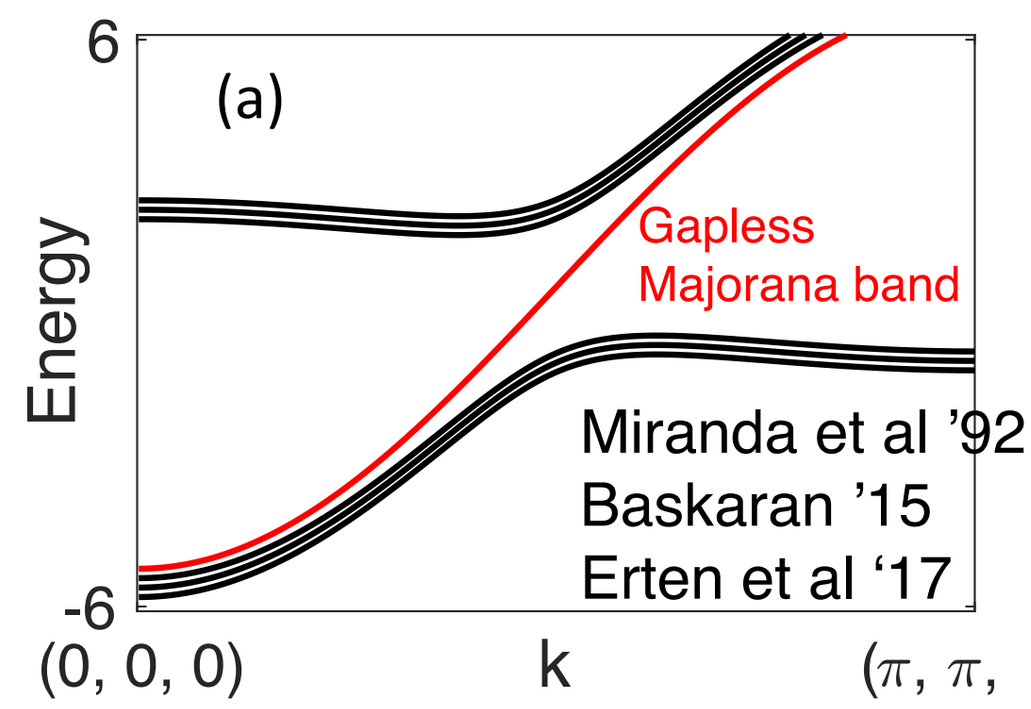
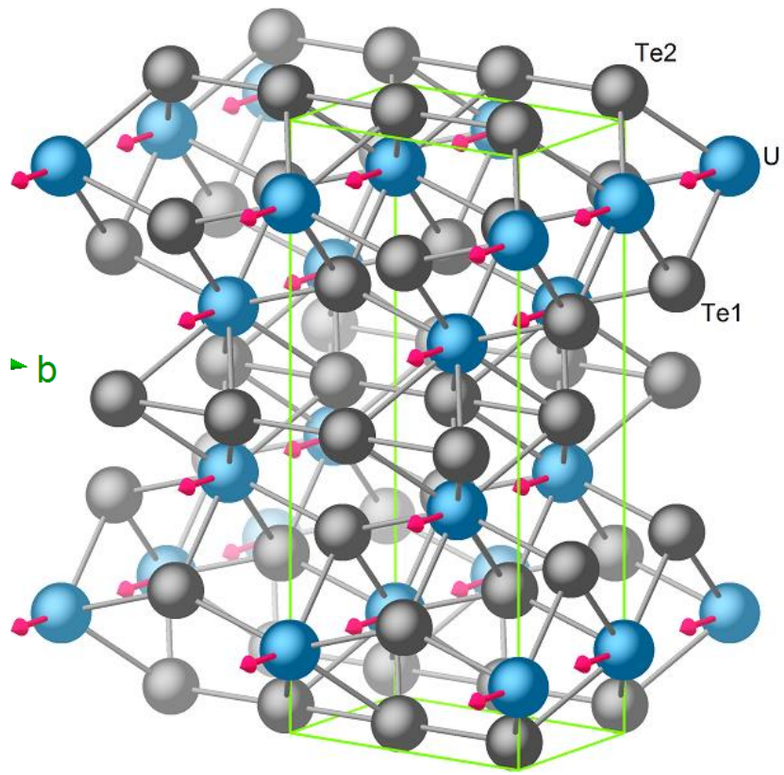


Last Thoughts

Last Thoughts

UTe_2



Ran et al, arXiv 1811.11808

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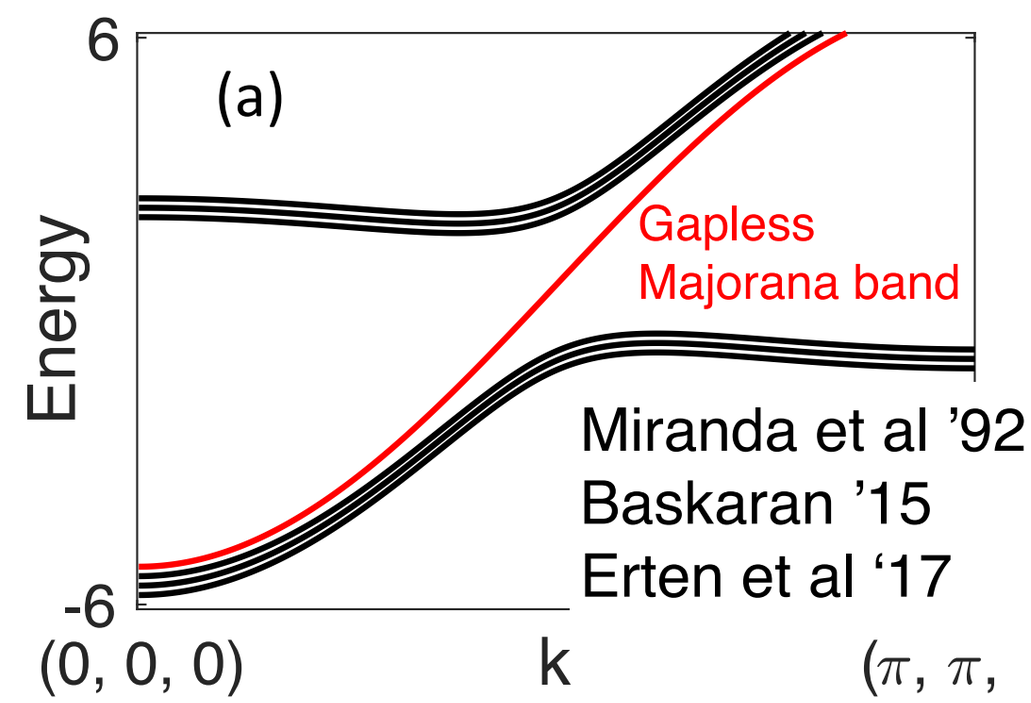

Miranda et al '92
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Last Thoughts

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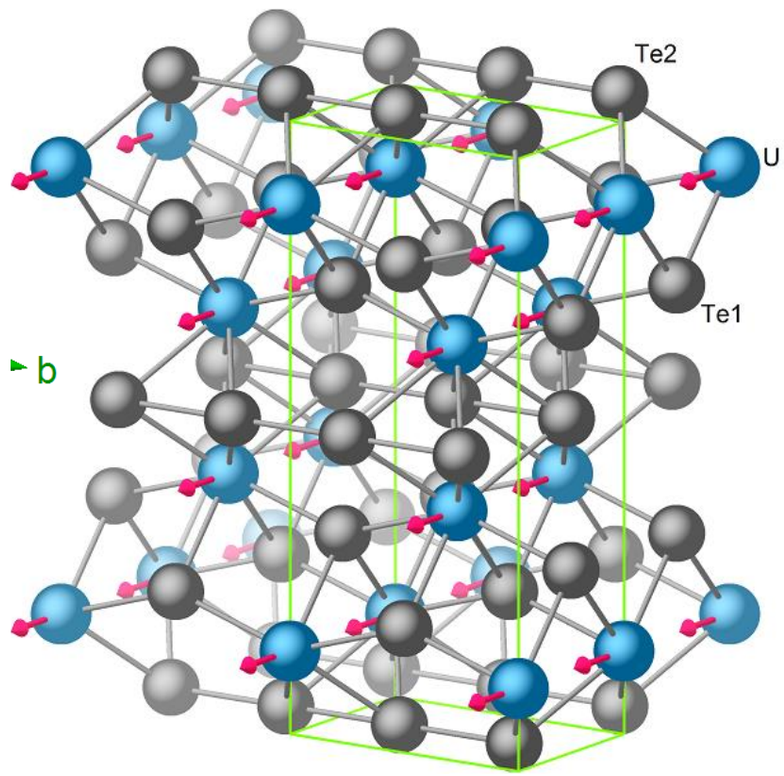

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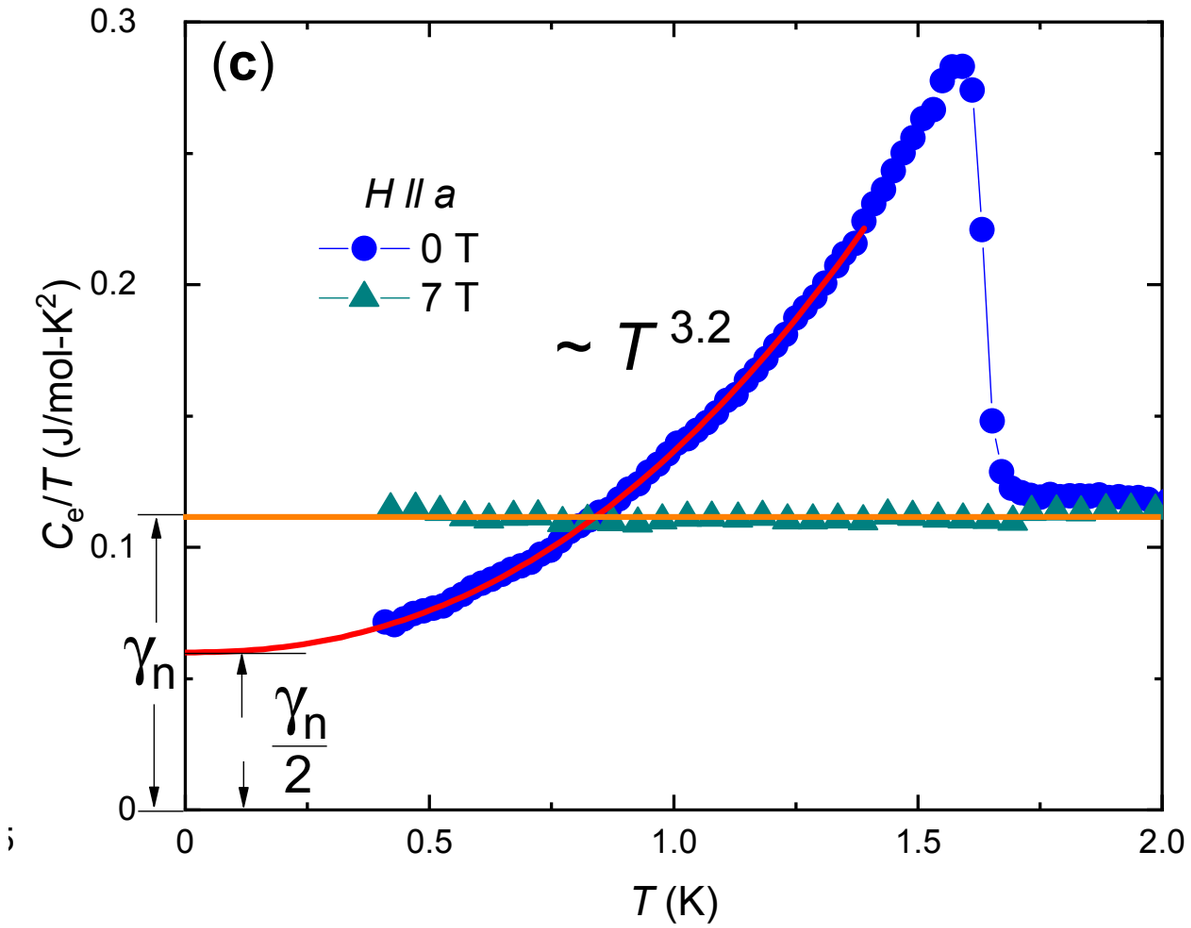
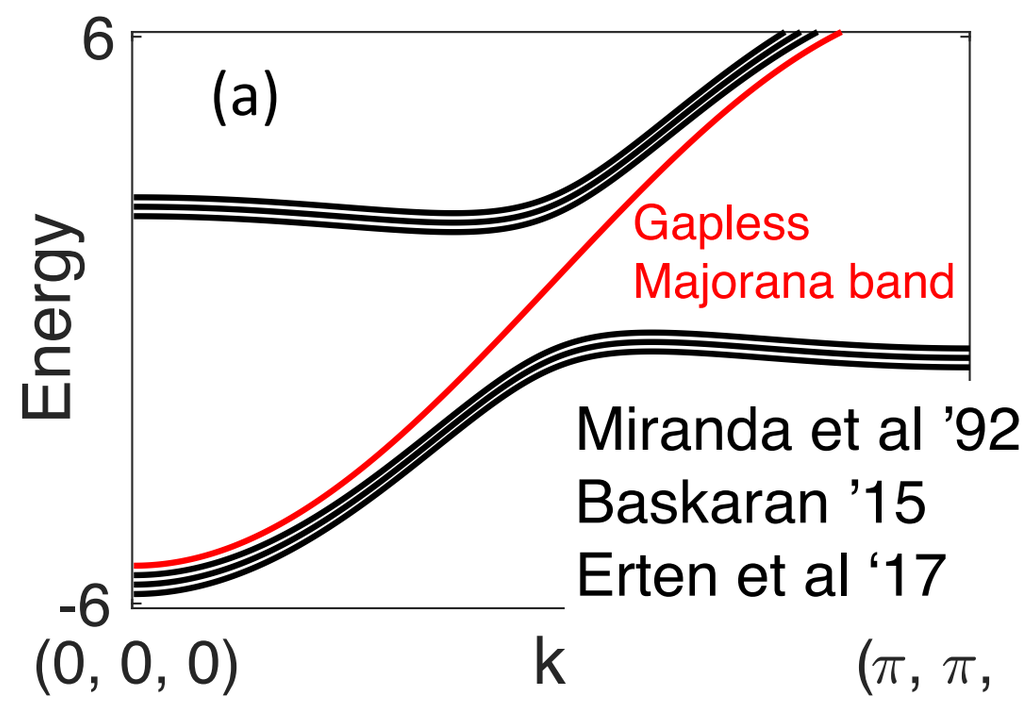
"Half gapped superconductivity"





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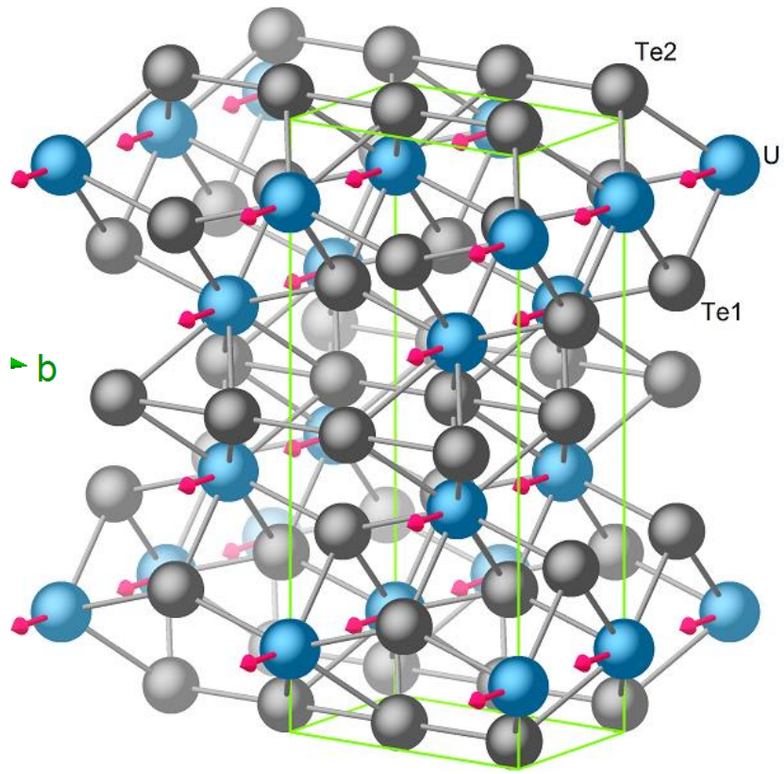

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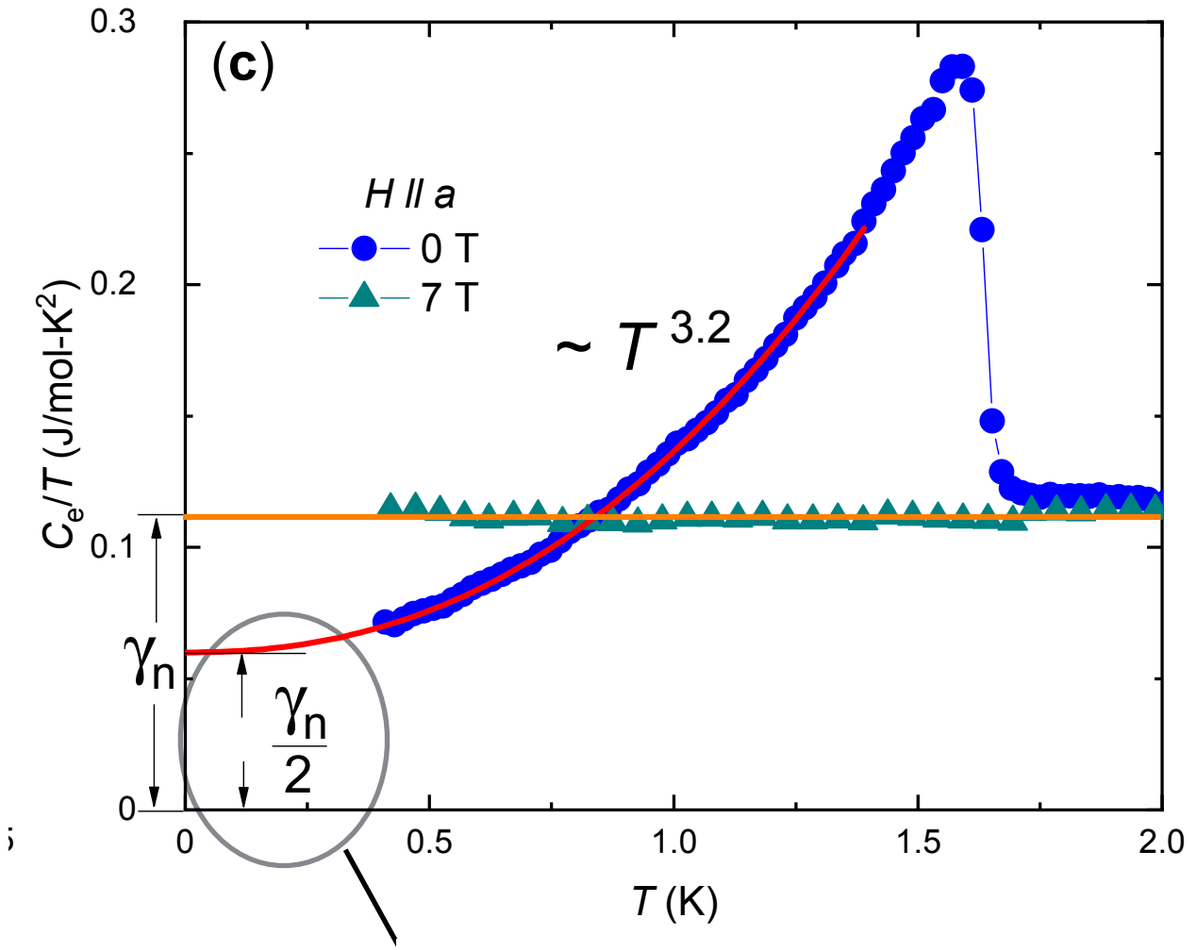

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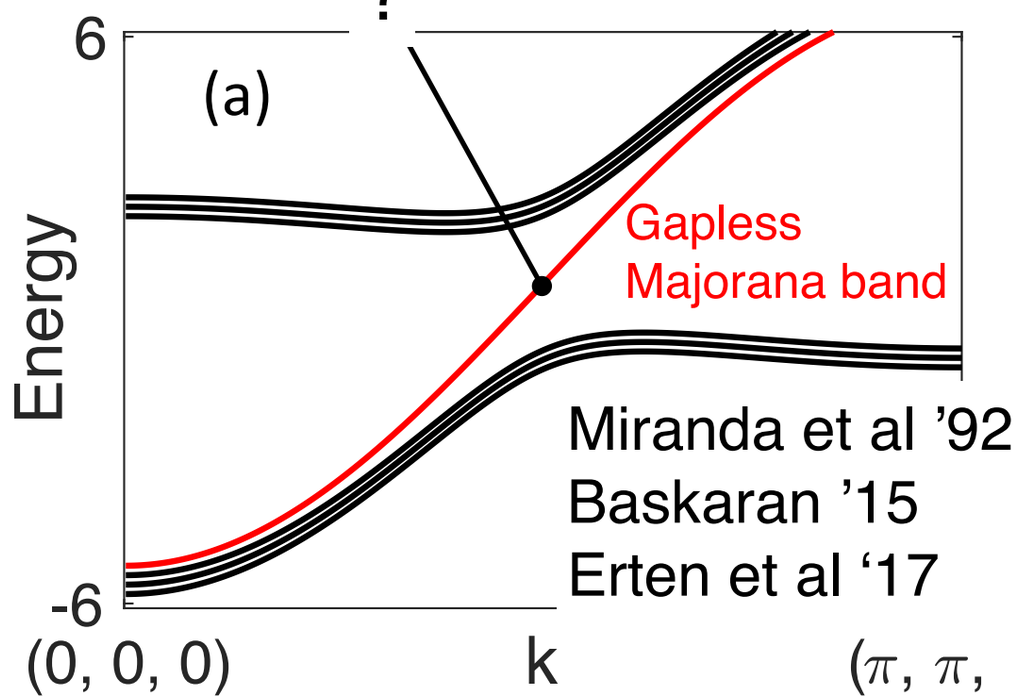

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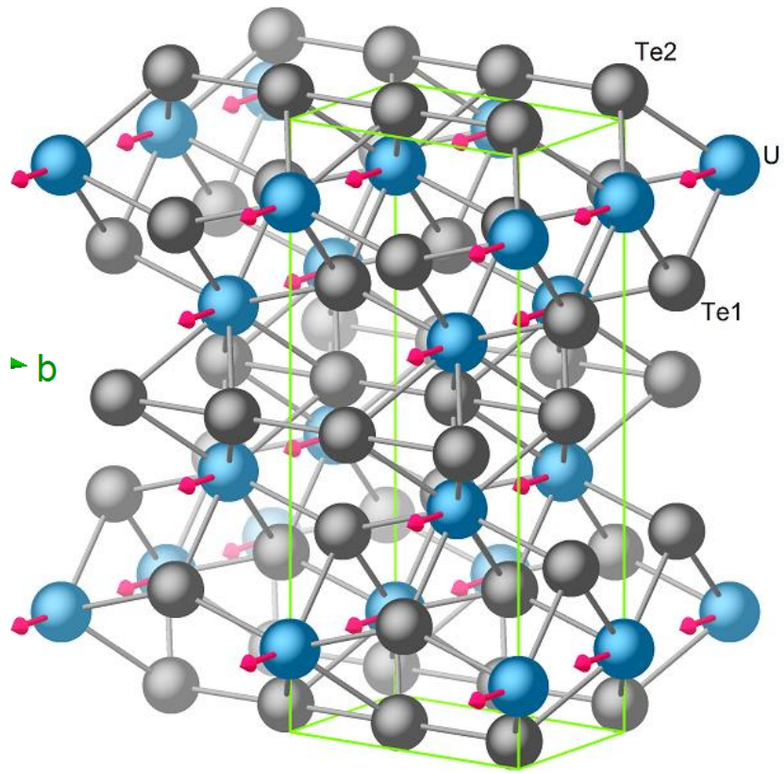

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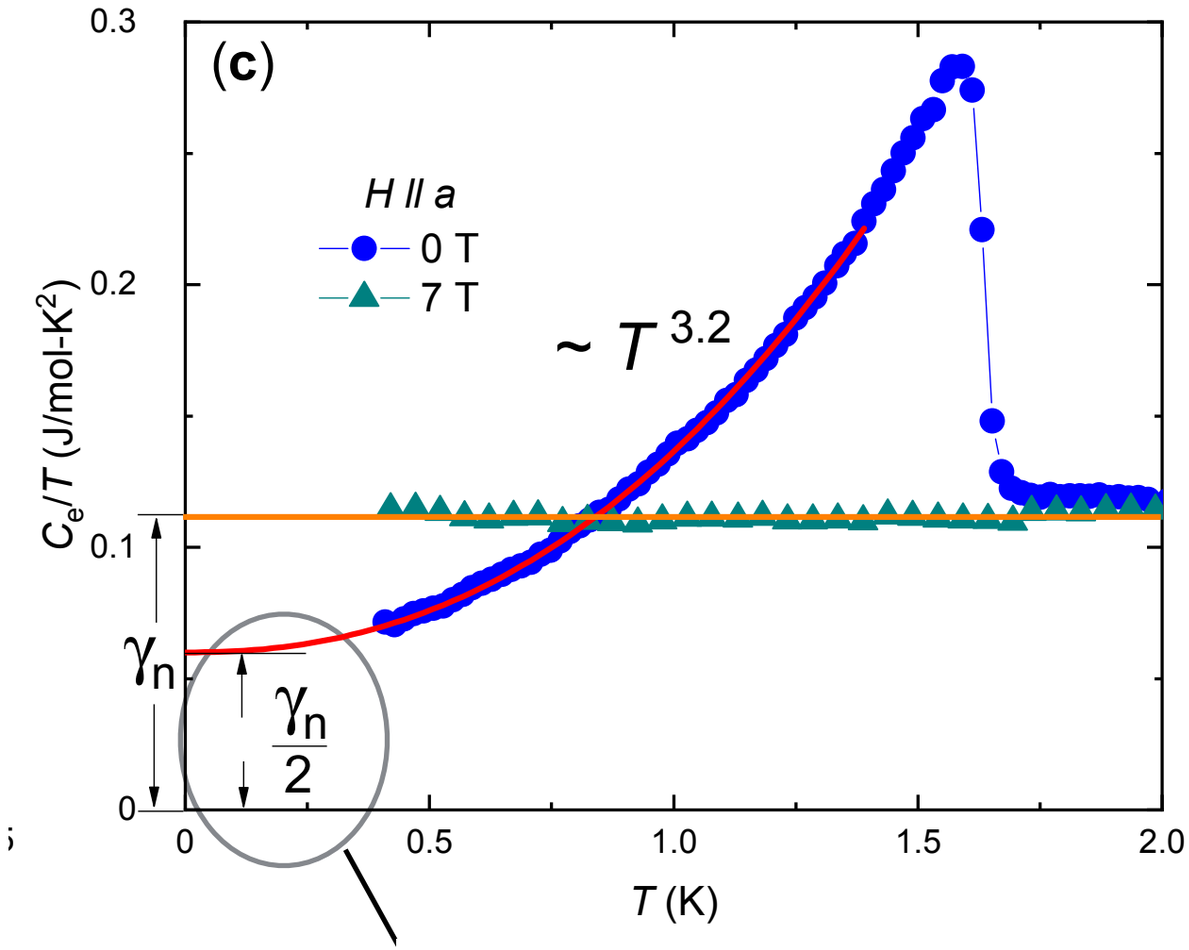

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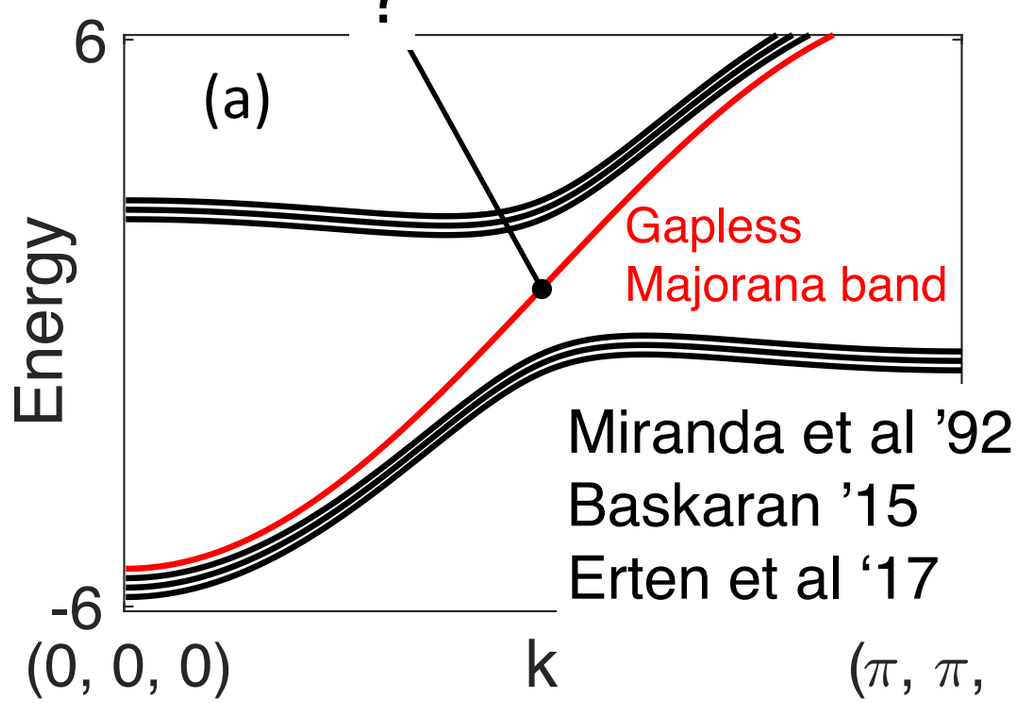

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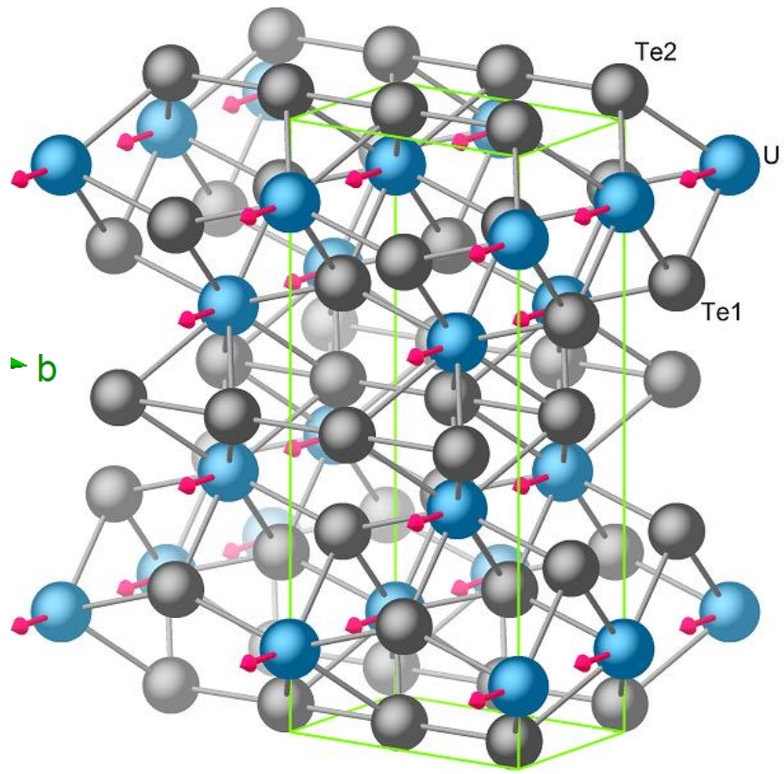

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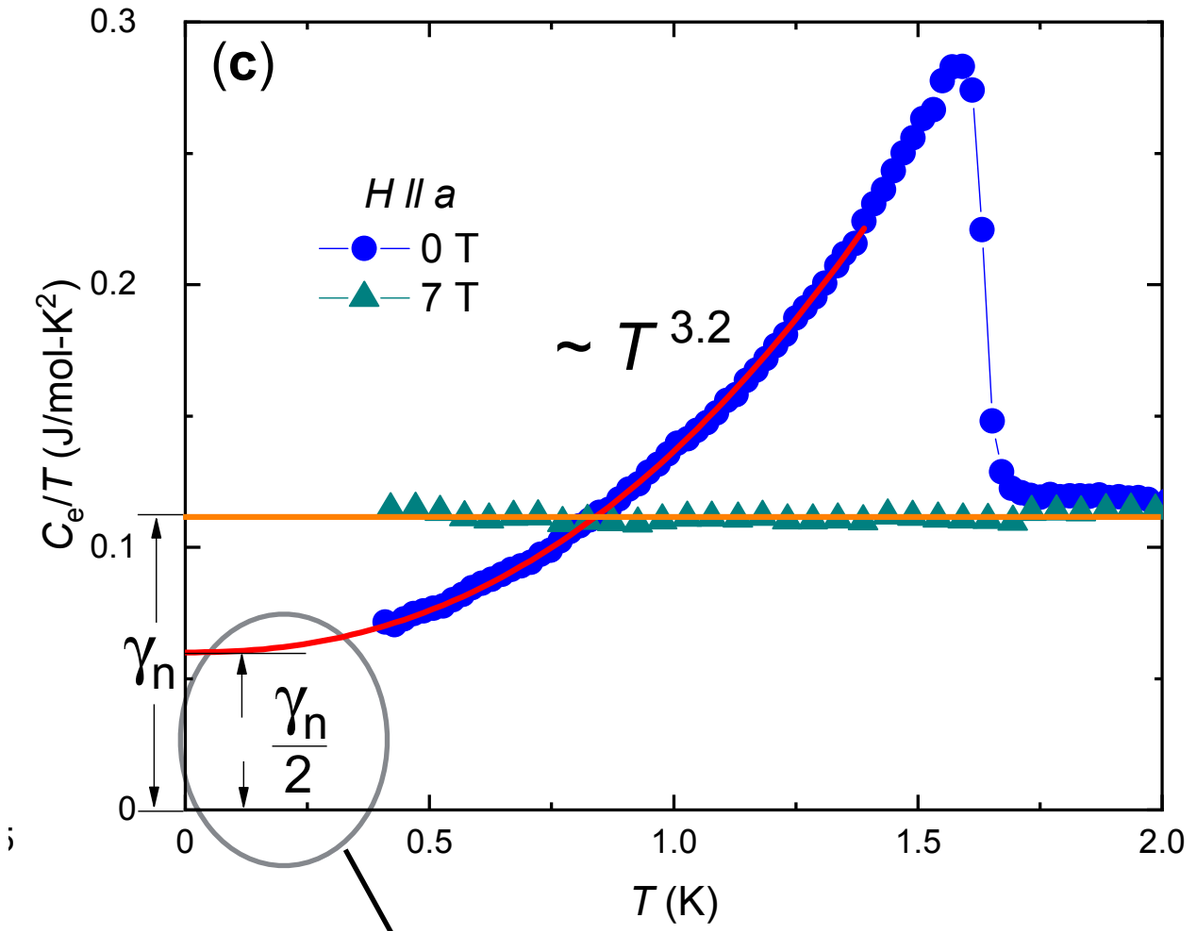

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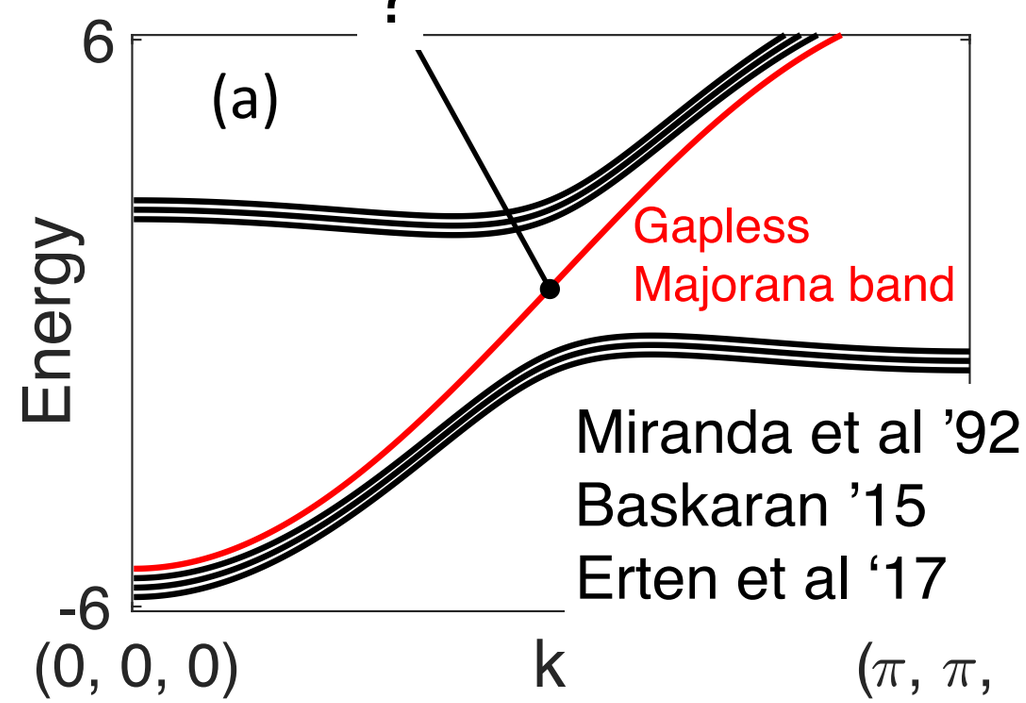
Projective nature of SC suggests fractionalized order



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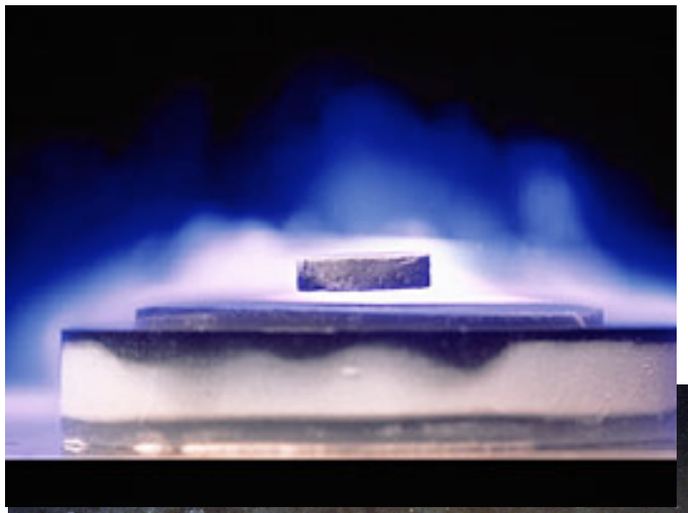
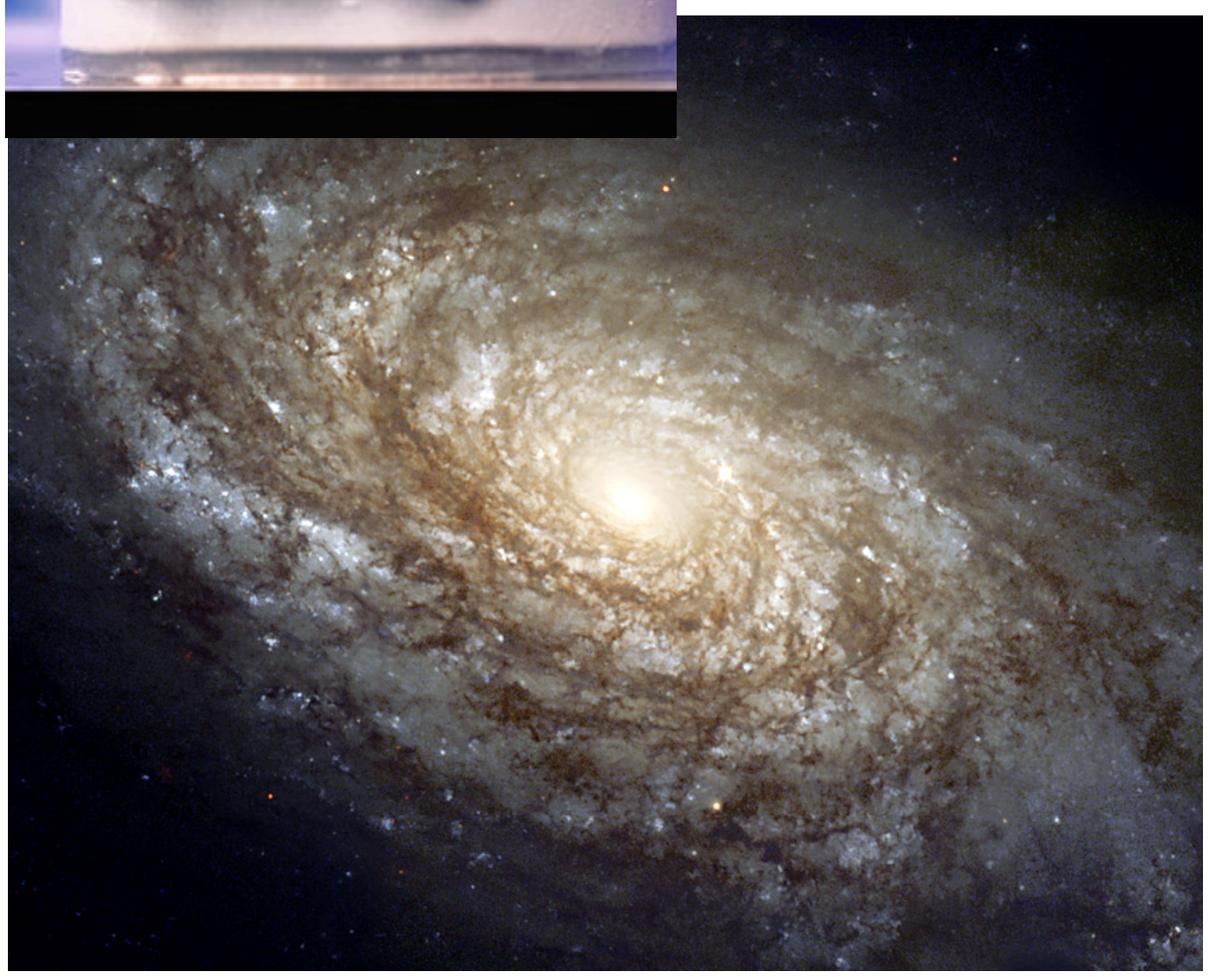
Cosmic Implications?





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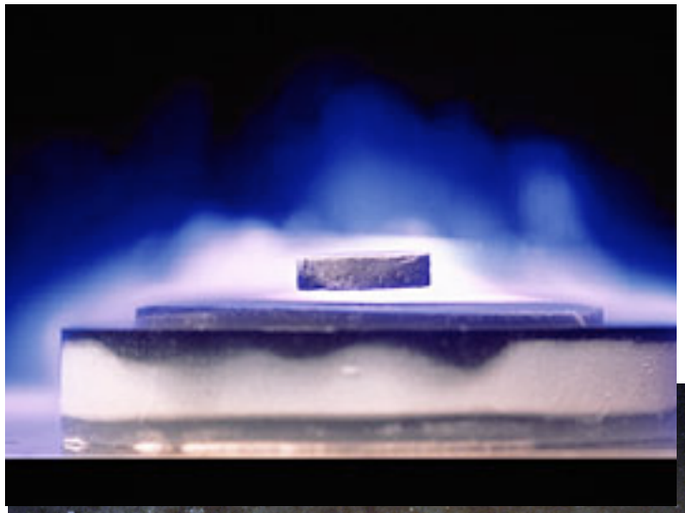

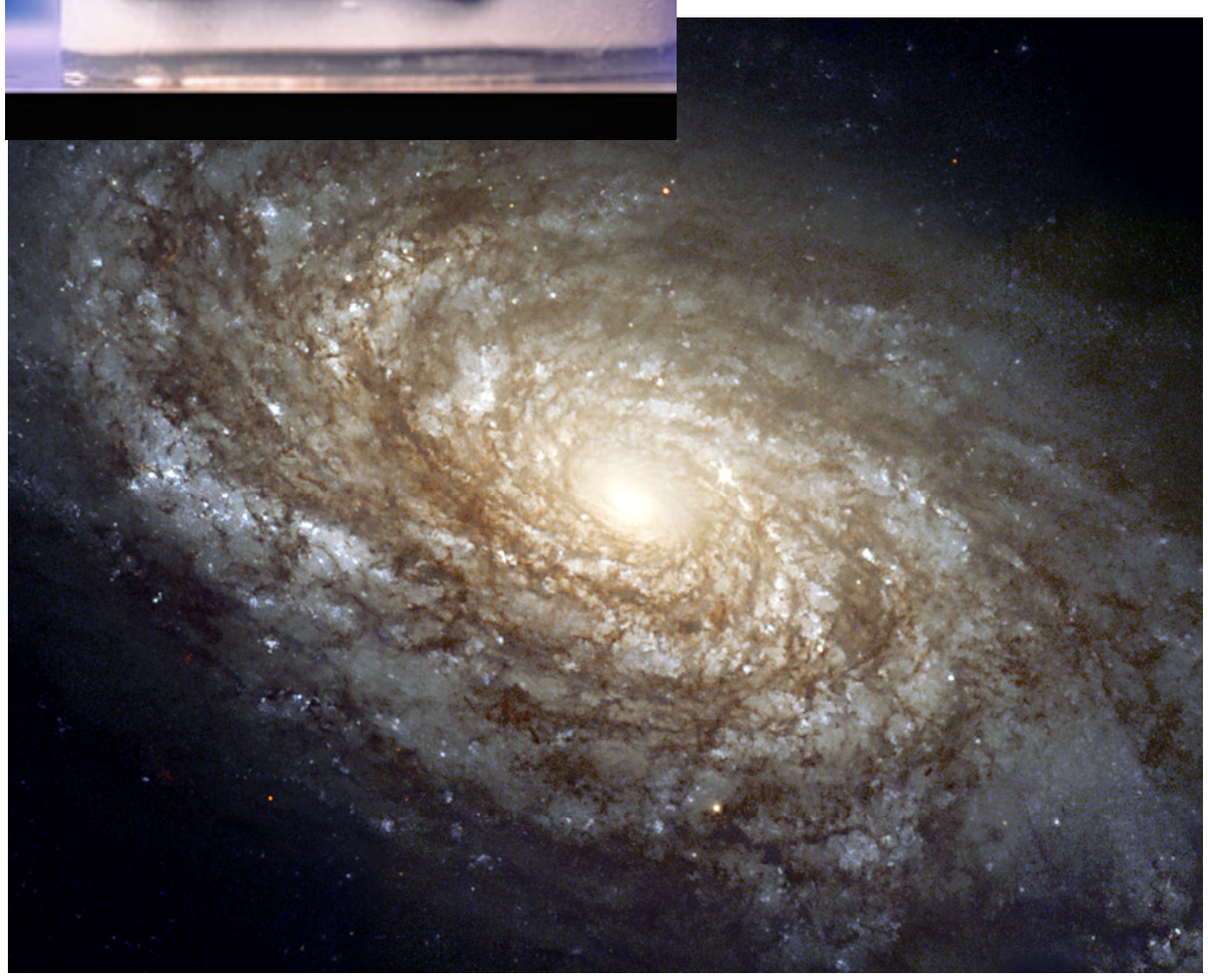

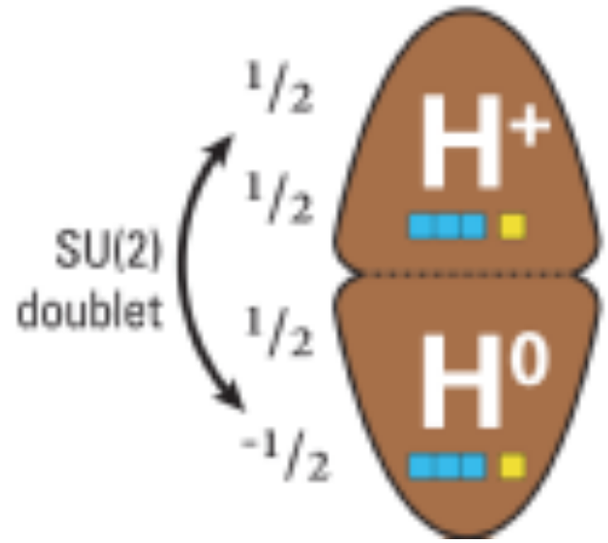

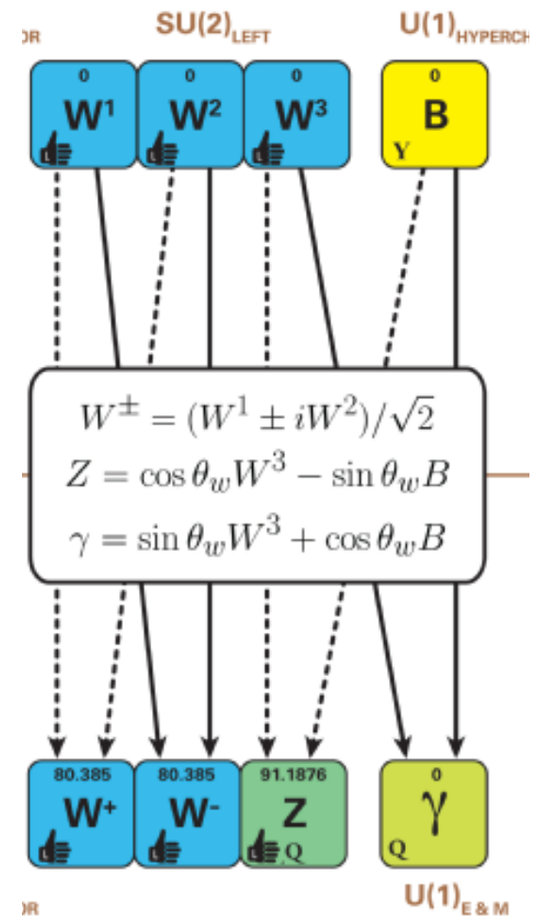

Higgs = an isospin 1/2 spinor

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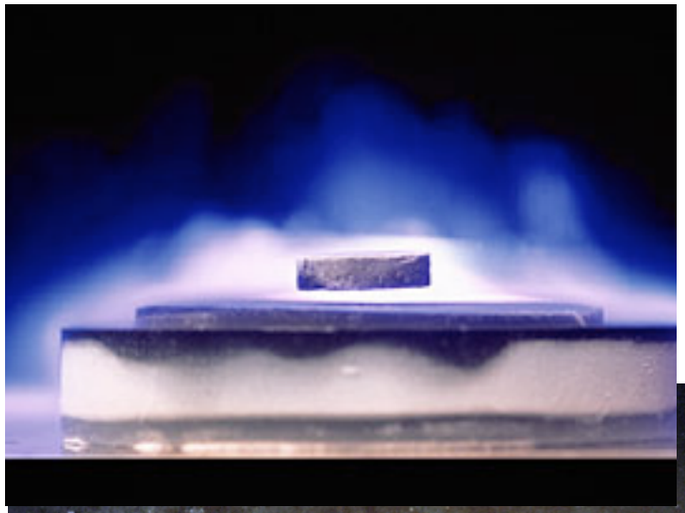

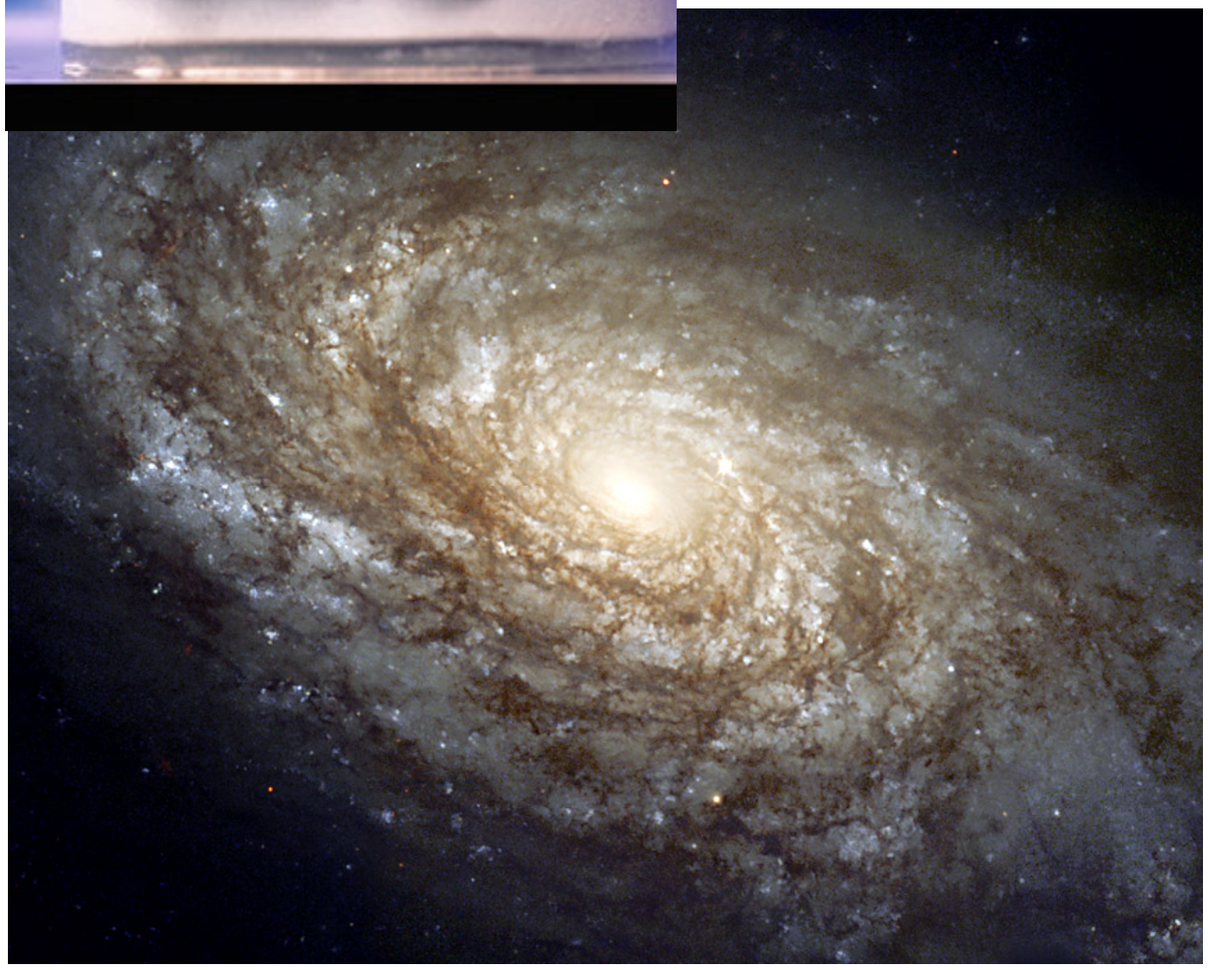

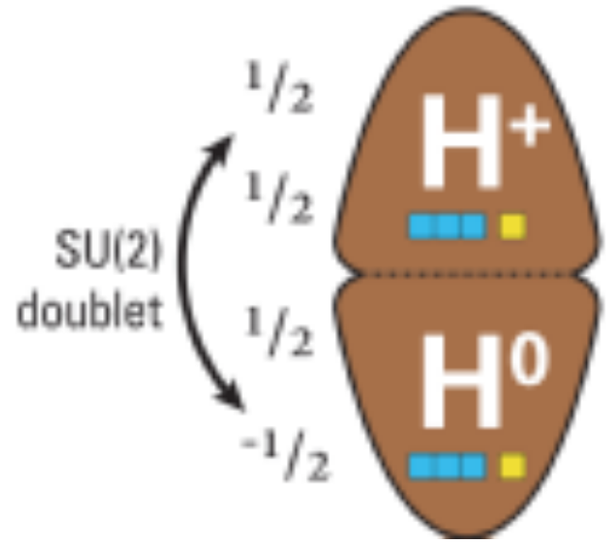

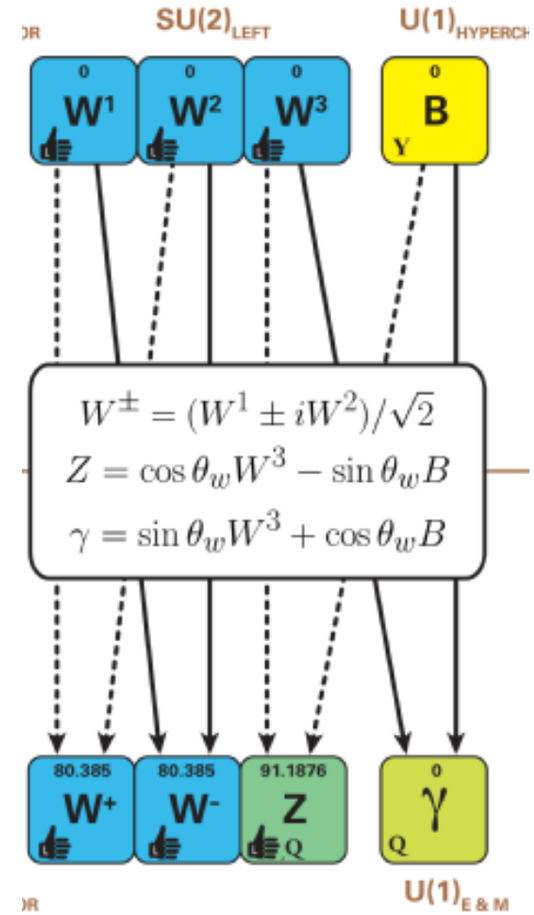

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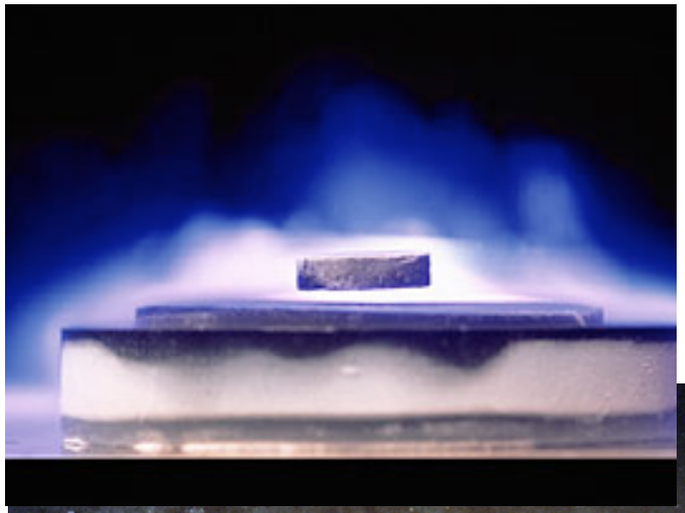

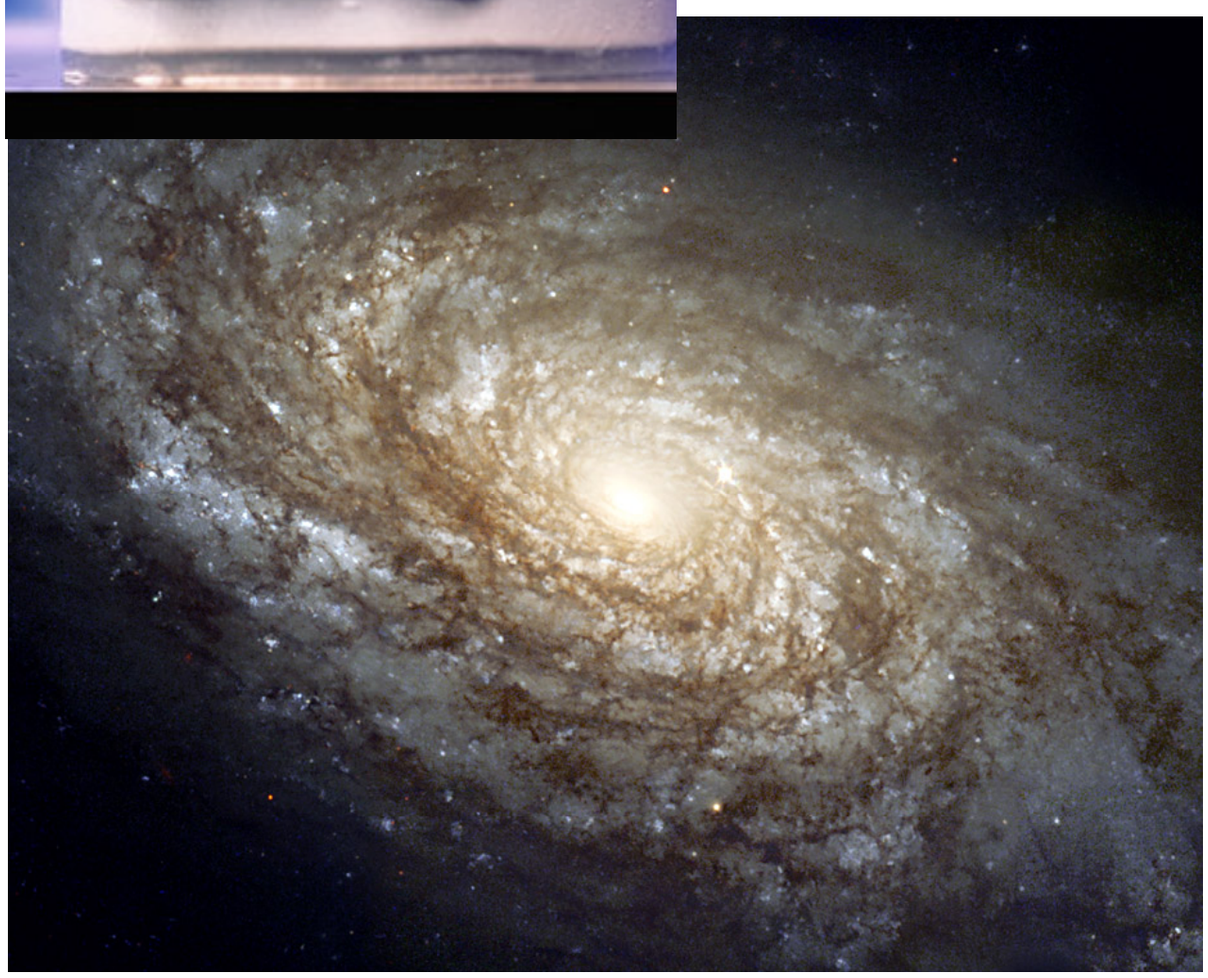

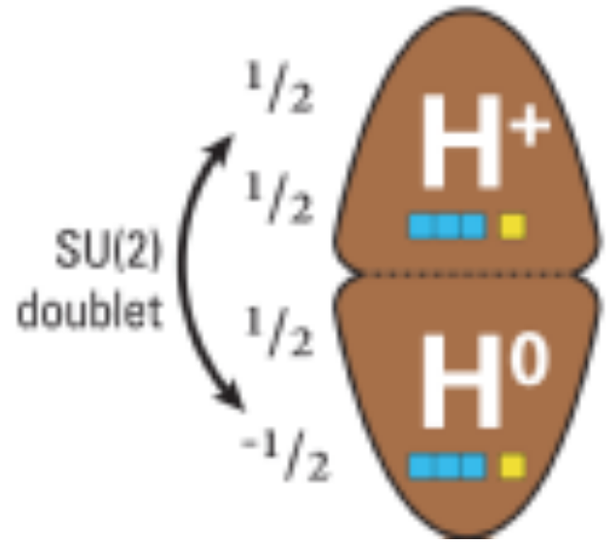

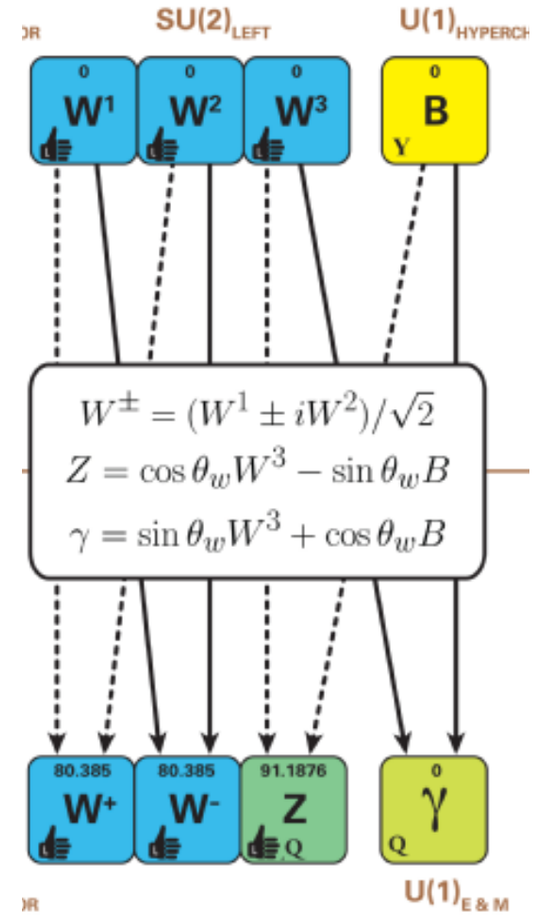

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Order fractionalization of three fermions?

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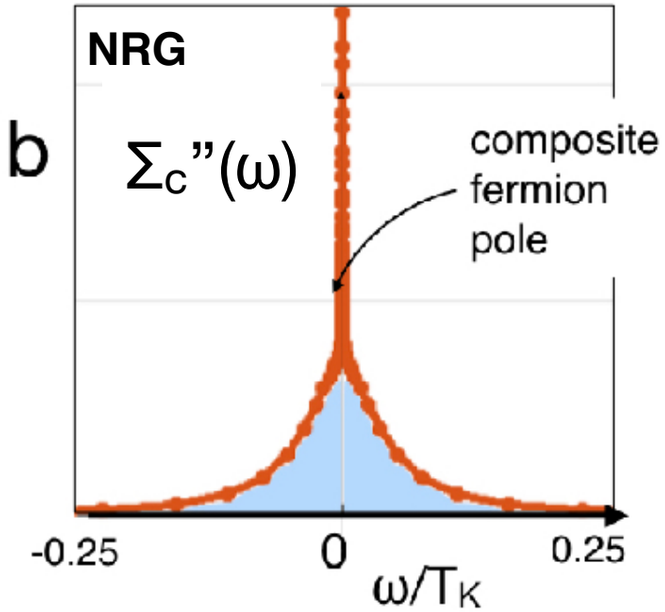

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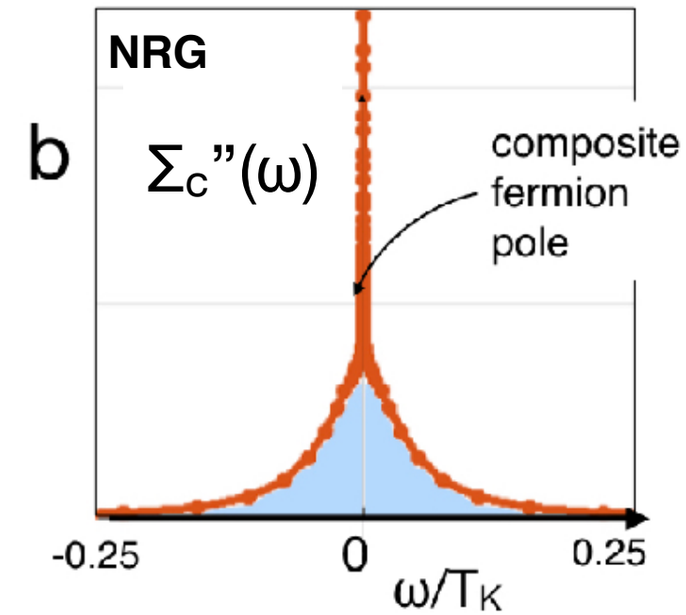

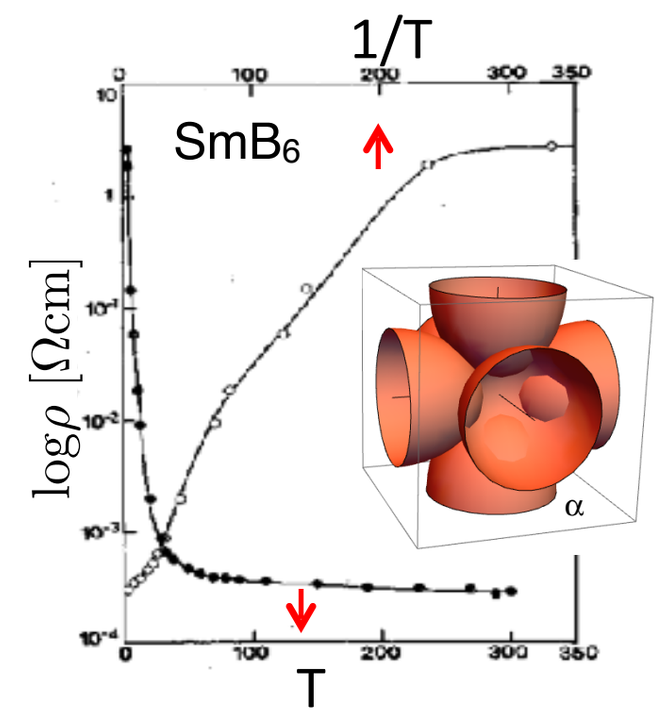

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Majorana
Fractionalization ?

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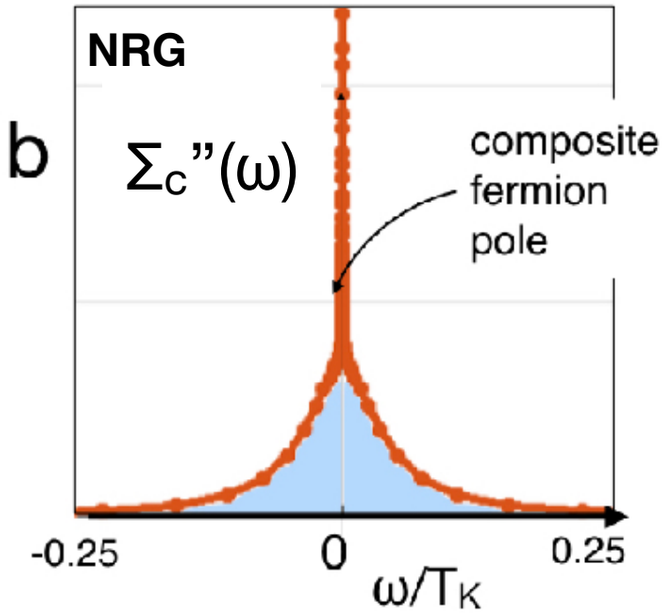

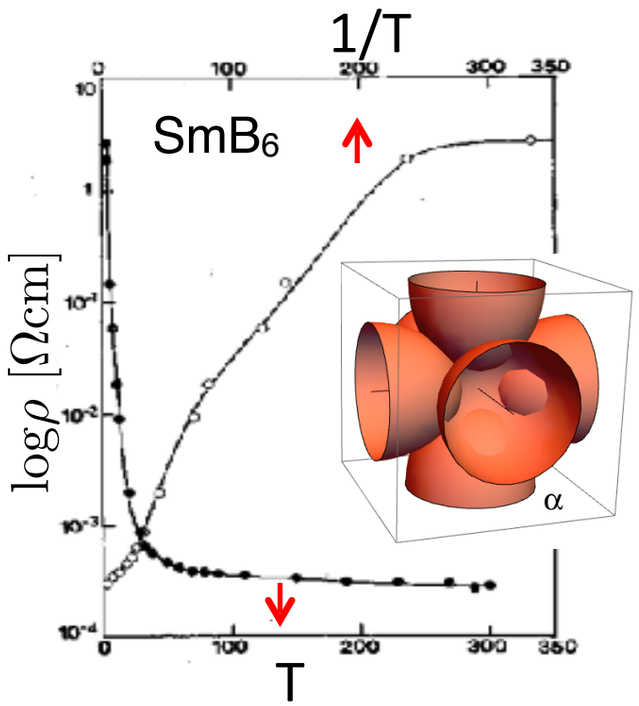

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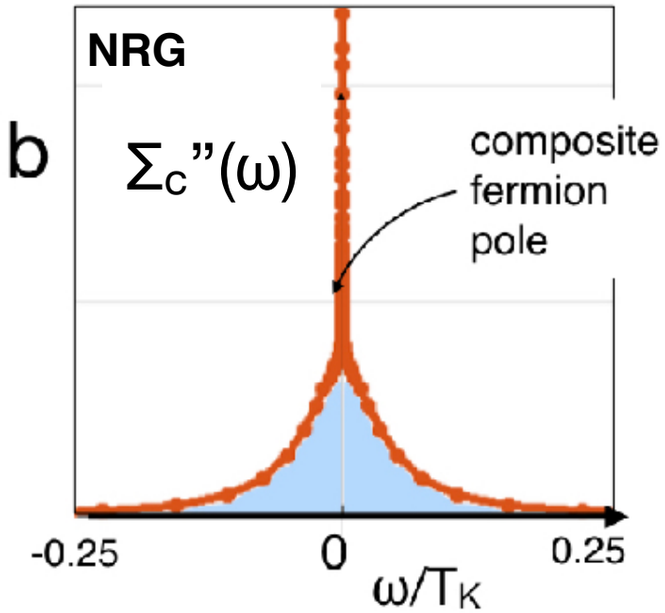

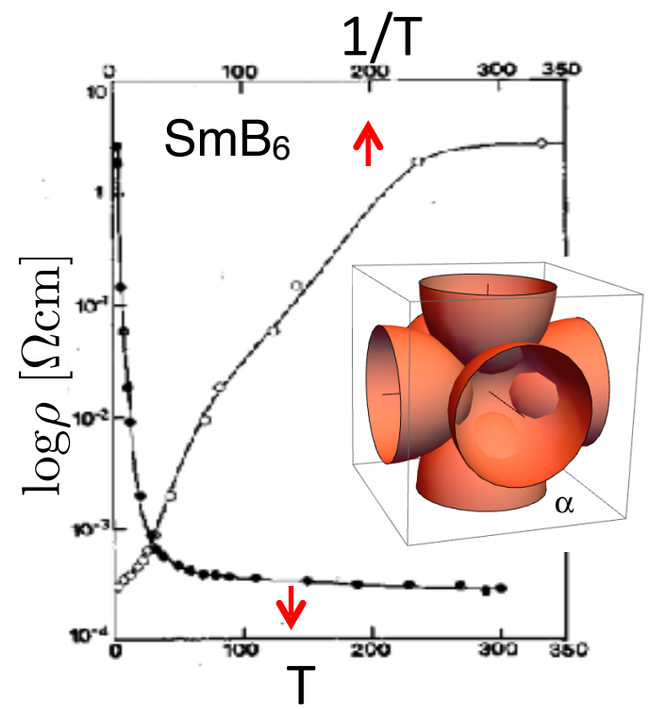

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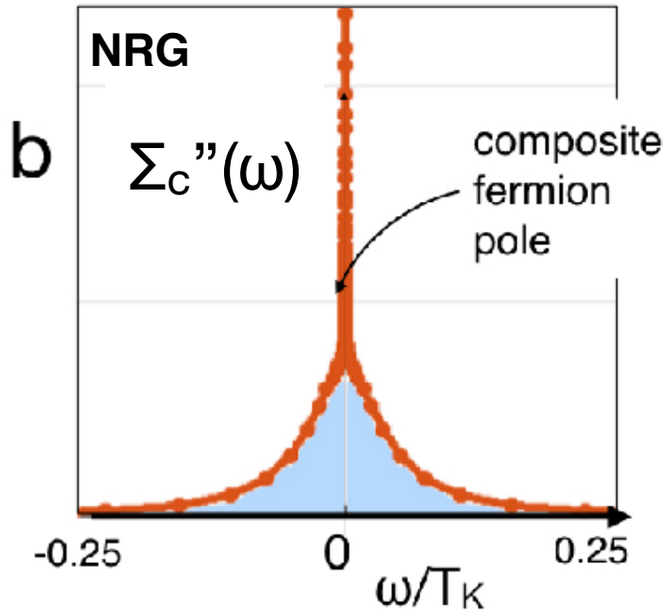

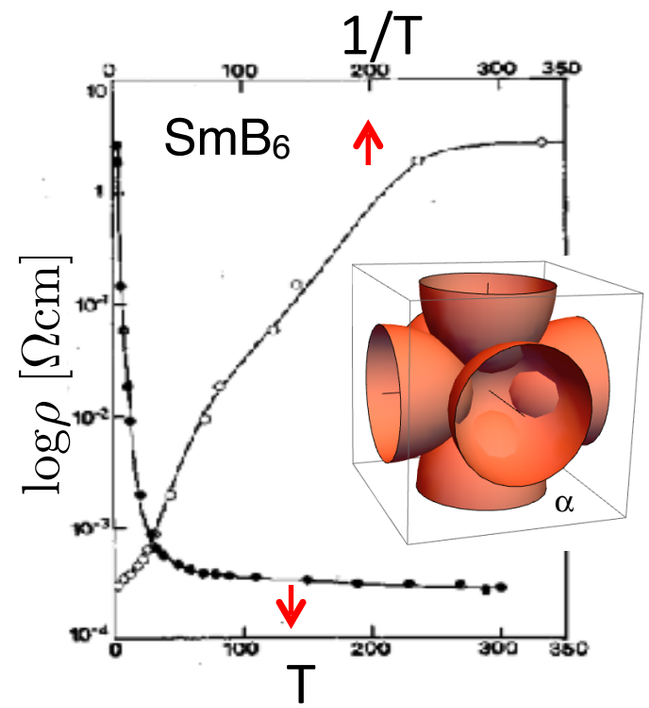

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$$\Sigma_{\lambda\lambda'}(2,1) \xrightarrow{|2-1|\to\infty} V_\lambda(2) V_{\lambda'}(1) g(2-1)$$

ODLRO in Space Time

$$\vec{S} = -\frac{i}{2}\vec{\eta}\times\vec{\eta}$$

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Thank You!